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Simplicial Models for the Global Dynamics of Attractors

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Given an unknown attractor $\mathscr{A}$ in a continuous dynamical system, how we can discover the topology and dynamics of $\mathscr{A}$? As a practical matter, how can we do so from only a finite amount of information? One way of doing so is to produce a semi-conjugacy from $\mathscr{A}$ onto a model system $\mathscr{M}$ whose topology and dynamics [illegible] tial features of this construction are that the model $\mathscr{M}$ can be explicitly described and that the finite amount of information needed to construct it is computable.

Key Words: attractor; Conley index; semi-conjugacy, simplicial complex.

1. INTRODUCTION

A natural problem in dynamics is to seek situations in which a finite amount of data (produced either numerically or analytically) allows the topology and dynamics of a compact invariant set S to be recovered, at least partially. Before considering results of this type, it is necessary to clarify exactly what it means for information about S "to be recovered." One point of view is to give a known system M and show that either M embeds in S, or that S maps via a semi-conjugacy onto M. The existence of a periodic orbit can be viewed as an example of the former (i.e., there is a semi-conjugacy $f\colon S^1 \to S$); while the conjugacy from the Smale horseshoe to shift dynamics is a classic example of the latter. In either point of view, the known topology and dynamics of M is transformed by the existence of the semi-conjugacy into information about S. The problem is to identify what model flow M admits such a semi-conjugacy.

If S is a compact invariant set in a continuous dynamical system, Conley's decomposition theorem [2] states that there is a semi-conjugacy

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from S onto a gradient-like system. This semi-conjugacy is formed by indentifying each component of the chain recurrent set to a point. While this is a powerful structure theorem, its practical utility is limited by the fact that the semi-conjugacy and the the gradient-like system are existential. The theorem gives no method for describing or understanding their structure, other than to first understand the structure of S itself. Obviously, if the goal of the analysis is to understand the global structure of S, this is not very useful.

Motivated by Conley's theorem, we seek a method to explicitly construct a compact space M, and explicitly define a flow on M, such that there is a surjective semi-conjugacy $f: S \to M$. The essential questions that must be addressed are:

- How much information about S is needed to construct the model flow and semi-conjugacy?
- How complicated can the model flow be?
- How do we guarantee surjectivity?

The first two questions are closely related: the complexity of the model is, in some sense, a measure of the information available about S. With no information about S, we can construct a semi-conjugacy onto a single (rest) point. With complete knowledge of S, we can construct a model flow which is conjugate to S.

It is important that we not only construct the model and semi-conjugacy, but that we also know exactly what the image of f in M is. It is only $\mathrm{im}(f)$ that carries information about S. Since $\mathrm{im}(f)$ is a compact invariant subset of M, if we can identify $\mathrm{im}(f)$, we can discard the rest of M. That is, the ability to identify $\mathrm{im}(f)$ is essentially equivalent to requiring f to be surjective. We will adopt this point of view, and one of the main features of this work will be to identify conditions that guarantee the surjectivity of f.

The information about the invariant set used to construct the model could take any number of forms: homological; measure-theoretic; a description of basic sets; etc. Following Conley's decomposition theorem, we develop a construction that requires information about a Morse decomposition of the invariant set: the complexity of the Morse sets themselves, and the complexity of their connecting information. These are measured by homological and combinatorial data: the Morse sets by their homology Conley indices; and the connection information by the connection matrix and the associated partial order.

The first theorem along this line was proved in [12]. There, the Morse decomposition consisted of a collection $M_0, M_1, ..., M_P$ with partial order $0 < 1 < \cdots < P$. The homology Conley index of M_i was assumed to be that of an orientable hyperbolic periodic orbit with unstable dimension $2i$ for $i < P$ and that of a hyperbolic fixed point with unstable dimension $2P$ for M_P. The Morse sets below M_P were each assumed to admit a Poincaré section, and some technical algebraic hypotheses were also assumed. From this information, a Morse–Smale flow on a $2P$ disk with P periodic orbits and one rest point, and a surjective semi-conjugacy to the disk, were constructed.

Similar results, with slightly simpler structures, were proved in [6, 7, 13, 15]. In [13], for example, the Morse decomposition consisted of $2P+1$ Morse sets $M_0^+, M_0^-, ..., M_{P-1}^+, M_{P-1}^-, M_P$, with partial order $(0, \pm) < (1, \pm) < \cdots < (P-1, \pm) < P$. The homology Conley indices were those of a hyperbolic fixed point of index p for $M_p^\pm$ and of index P for M_P. The connection matrix was assumed to have $\Delta_{qp} \neq 0$ if and only if p and q are adjacent in the partial order. With this structure, a Morse–Smale flow on a P-disk with $2P+1$ rest points and no periodic orbits, and a surjective semi-conjugacy to the disk, were constructed.

In a similar vein, the conjugacy from the Smale horseshoe to shift dynamics has been generalized[1, 16]. These can be paraphrased: given an isolated invariant set in a discrete system, whose Conley index behavior "looks like" that of a Smale horseshoe, there is a semi-conjugacy from the invariant set onto shift dynamics.

All of the results for continuous systems are similar both in the nature of their hypotheses, and in the manner in which the model flow and semi-conjugacy are constructed. They are all fairly restrictive in their hypotheses, and might be thought of as examples of some more general theorem on the existence of semi-conjugacies. We seek here to formulate such a theorem. Its statement will involve the terminology of the Conley index theory, which is reviewed in Section 2. To state our main results, we make the following assumptions on the invariant set.

(**H0**) $\mathscr{A}$ is an attractor in a continuous semi-flow on a locally compact metric space X. On $\mathscr{A}$ itself, there is a complete two-sided flow.

(**H1**) $\mathscr{A}$ has a Morse decomposition $\{S_p\}_{p \in P}$ indexed by the partially ordered set $(P, <)$.

(**H2**) Each Morse set S_p has the homology Conley index of a hyperbolic rest point. That is, for each $p \in P$, there exists an $n(p)$ such that

$$CH_k(S_p) \cong \begin{cases} Z, & k = n(p) \\ 0, & \text{otherwise.} \end{cases}$$

(**H3**) There is a unique connection matrix $\Delta(P)$. This matrix has the property that Morse sets S_p and S_q are adjacent in the flow-defined ordering if and only if the connection matrix entry Δ_{qp} is an isomorphism.

From the partial order $(P, <)$, we can construct in a natural way a simplicial complex $\mathscr{M}(P, <)$ by creating a simplex for every totally ordered chain in P. As we will see in Section 3, this simplicial complex admits a flow ψ: $\mathscr{M} \times \mathbb{R} \to \mathscr{M}$ which leaves each simplex invariant and has the vertex set $\{M_p\}_{p \in P}$ as a Morse decomposition. This will be the model flow that is the target of the semi-conjugacy from $\mathscr{A}$. Its crucial feature is that it is constructed directly from the partial order $(P, <)$—no further information about the topology or dynamics of $\mathscr{A}$ is required.

THEOREM 1.1. *Suppose* $\mathscr{A}$ *is an attractor with flow* ϕ *satisfying* (**H0**)–(**H3**). *Let* $\mathscr{M}(P, <)$ *be the simplicial complex generated by the poset* $(P, <)$. *Then, up to a time reparameterization of* ϕ, *there is continuous semi-conjugacy* f: $\mathscr{A} \to \mathscr{M}(P, <)$. *That is, there is a function* θ: $\mathscr{A} \times \mathbb{R} \to \mathbb{R}$ *which is monotone increasing in* t *for every* $x \in \mathscr{A}$, *such that* $f \circ \phi(x, \theta(x, t)) = \psi(f(x), t)$.

The time reparameterization is a technicality, and is only introduced to guarantee that if $f(x) = f(y)$, then $f(x \cdot t) = f(y \cdot t)$. The time reparameterization does not change any of the essential dynamical features of the flow on $\mathscr{A}$, so it is not too imprecise to interpret this theorem as "there is a semi-conjugacy from $\mathscr{A}$ to $\mathscr{M}$."

This theorem does not guarantee that the semi-conjugacy is surjective. At this point, it is not clear whether this is a technical shortcoming of the proof, or whether there are examples in which (**H0**)–(**H3**) do not produce surjectivity. It is also natural to ask if the model reproduces the Conley index information used to construct it. That is, since $\mathscr{M}(P, <)$ has a Morse decomposition with the same flow-defined ordering, does it also have the same Conley indices for the Morse sets? Does it have the same connection matrix? Is f_* a conjugacy between the algebra on $\mathscr{A}$ and the algebra on $\mathscr{M}$?

It turns out that the two questions are closely related. Our proof of surjectivity will use the homology Conley index, and it might be conjectured that, if M_p and S_p have the same homology Conley index for all p, then f is surjective. While we cannot prove such a relationship at this point, we can formulate a condition which is very close in spirit to "M_p and S_p have the same homology Conley index" and which implies both the equivalence of the indices and the surjectivity of f.

For every $p \in P$, let $A_p = \{q \in P \mid q < p\}$, and let $M(A_p)$ be the subcomplex of $\mathscr{M}(P, <)$ spanned by vertices in A_p. We will see in Section 3 that $M(A_p)$ is a homogeneous $(n(p) - 1)$-complex. We will be interested in the following special case:

(**H4**) For every $p \in P$, the complex $M(A_p)$ is homeomorphic to the $(n(p) - 1)$-sphere.

As we will see in Theorem 3.15, there is a simple combinatorial condition that implies (**H4**)—a condition that is satisfied by all of the motivating examples in [6, 7, 13, 15]. The constructions of simplicial models in those examples are then subsumed into the following:

THEOREM 1.2. *If $\mathscr{A}$ is a compact attractor with flow ϕ that satisfies* (**H0**)–(**H4**), *then*

(1) *The semi-conjugacy $f: \mathscr{A} \to \mathscr{M}(P, <)$ is surjective.*

(2) *For every interval $I \subset P$, $f_*(I): CH_*(\mathscr{A}(I)) \to CH_*(M(I))$ is an isomorphism.*

(3) *The Morse decomposition $\{M_p\}_{p \in P}$ has a unique connection matrix $\Delta_M(P)$, which is conjugate to $\Delta(P)$ via the isomorphism $F = \bigoplus_{p \in P} f_{p^*}$: $\bigoplus_{p \in P} CH_*(S_p) \to \bigoplus_{p \in P} CH_*(M_p)$. That is, $\Delta_M(P) \circ F = F \circ \Delta(P)$.*

There are several important features to these results. First, the hypotheses are verifiable in practice, with only a minimal amount of information about $\mathscr{A}$ required to carry out the computations. Second, once the partial order $(P, <)$ is known, the complex $\mathscr{M}(P, <)$ can be easily constructed (and property (**H4**) checked) without further knowledge of $\mathscr{A}$ or its flow required. This ability to construct and explore $\mathscr{M}$ is of central importance. Once constructed, $\mathscr{M}$ serves as a model for the flow on $\mathscr{A}$. If the semi-conjugacy is surjective, then the complexity of $\mathscr{M}$ (both in its topology and dynamics) serves as a lower bound for the complexity of $\mathscr{A}$. In sum, a finite amount of information about $\mathscr{A}$ allows a model flow $\mathscr{M}$ to be explicitly constructed, and to guarantee that the dynamical structures revealed by that model will be a lower estimate for the dynamics on $\mathscr{A}$.

The paper is organized as follows. The next section briefly reviews the relevant material of the Conley index theory. Section 3 constructs $\mathscr{M}(P, <)$ and examines its topological and dynamical properties. Section 4 constructs the semi-conjugacy f, and Section 5 gives the proof of Theorem 1.2. The paper closes with a discussion of verifiability and necessity of the hypotheses, and the possible improvements and extensions of the results.

2. THE CONLEY INDEX

We begin with a brief review of the relevent portions of the Conley index theory. The basic references for this material are [2, 5, 10, 11, 14, 22, 23]. The Conley index was introduced to study *isolated invariant sets*. An invariant set S is an isolated invariant set if there is a compact neighborhood N of S such that S is the maximal invariant set in N. The neighborhood N is an *isolating neighborhood* for S. The Conley index of S

studies the nature of the flow around S, rather than on S itself. The features of the index theory that we will be concerned with are the homology Conley index, Morse decompositions and the connection matrices that the index theory uses to study them, the behavior of the index under semi-conjugacies, and the dependence of the index on the ambient space. Our blanket assumption will be that X is a locally compact metric space with a semi-flow, and $\mathscr{A} \subset X$ is a compact attractor (with a complete flow).

2.1. *Morse Decompositions.* While we assume that $\mathscr{A}$ is an attractor in X, this does not mean that the dynamics on $\mathscr{A}$ is chain-recurrent. Thus, Conley's decomposition theorem implies that $\mathscr{A}$ may admit a further decomposition–a Morse decomposition. The simplest form of a Morse decomposition is an *attractor-repeller decomposition.* If S is an isolated invariant set, $A, R \subset S$, then the pair (A, R) is an attractor-repeller pair in S if

(1) A is an attractor in S: there is a positively invariant neighborhood U of A in S with $\omega(U) = A$.

(2) R is the dual repeller to A in S: $R = S \setminus \{x \mid \omega(x) \subset A\}$.

Note that A and R are both isolated invariant sets, and if

$$C(R, A) = \{x \in S \mid \omega^*(x) \subset R,\ \omega(x) \subset A\},$$

then $S = R \cup C(R, A) \cup A$. That is, an attractor-repeller pair gives a decomposition of S into (two) finer invariant sets and connecting orbits between them.

More generally, a *Morse decomposition* is a decomposition of an invariant set into a finite number of invariant subsets (i.e., Morse sets) and connecting orbits between them. That is, a Morse decomposition of S consists of a finite collection of isolated invariant subsets S_p, indexed by some set $\mathscr{P}$, with a partial order $<$ on $\mathscr{P}$. The requirement is that, if $x \in S \setminus \bigcup_{p \in \mathscr{P}} S_p$, then there exist $q < p$ such that $x \in C(S_p, S_q)$. That is, the partial order must respect the flow: orbits can only flow "down" through the partial order. A partial order on $\mathscr{P}$ which respects the flow is referred to as an *admissible* partial order. The most natural way to produce an admissible partial order is to let the flow generate it. Set $q < p$ if $C(S_p, S_q) \neq \emptyset$, and take the transitive closure. We refer to this as the *flow-defined partial order*, and any admissible order must be a refinement of it.

If $\{S_p\}_{p \in \mathscr{P}}$ is a Morse decomposition of S, then each S_p is an isolated invariant set. S contains more isolated invariant sets, some of which are can be produced by $(\mathscr{P}, <)$ as follows. A subset $I \subset \mathscr{P}$ is an *interval* in $\mathscr{P}$ if $r \in I$ whenever $p < r < q$ and $p, q \in I$. Disjoint intervals I and J are ordered $I < J$ if $i < j$ for every $i \in I$, $j \in J$; they are adjacent if $IJ = I \cup J$ is also an interval (i.e., if no element of $\mathscr{P}$ lies "between" I and J). If I is an interval, let $S(I) = \bigcup_{i \in I} S_i \cup \bigcup_{i,\, j \in I} C(S_j, S_i)$. Then each $S(I)$ is an isolated invariant set, and if I and J are adjacent intervals with $I < J$, then $(S(I), S(J))$ is an attractor-repeller pair for $S(IJ)$.

An interval $I \subset P$ is an *attracting interval* if $p \in I$ and $q < p$ implies $q \in I$. If I is an attracting interval, then $S(I)$ is an attractor in S. We will be particularly interested in two types of attracting intervals. For every $p \in P$, we define $A_p = \{q \in P \mid q < p\}$ and $A_p^+ \{q \in P \mid q \leqslant p\}$. Then A_p and $\{p\}$ are adjacent intervals, and $(S(A_p), S_p)$ is an attractor-repeller decomposition of $S(A_p^+)$.

2.2. *The Homology Conley Index.* These are the structures the Conley index studies. The Conley index of an isolated invariant set is defined in terms of an *index pair*: a compact pair (N, L) such that

(1) $\overline{N \backslash L}$ is an isolating neighborhood for S.

(2) L is positively invariant in N: if $x \in L$ and $x \cdot [0, T] \subset N$, then $x \cdot [0, T] \subset L$.

(3) L is an exit set for N: if $x \in N$ and $x \cdot T \notin N$, then there is a $0 < t < T$ such that $x \cdot [0, t] \subset N$ and $x \cdot t \in L$.

An index pair is further said to be *regular* if, in addition, the function $\varpi\colon N \to [0, \infty)$ defined by

$$\varpi(x) = \begin{cases} \sup\{t > 0 \mid x \cdot [0, t] \subset N \backslash L\} & \text{if } x \in N \backslash L \\ 0 & \text{if } x \in L \end{cases}$$

is continuous. Observe that this implies that for a regular index pair L is a neighborhood deformation retract (along flow lines) in N. Index pairs (indeed, regular index pairs) always exist, and the homotopy type of the quotient space N/L is independent of the index pair chosen. It is that homotopy type which defines the Conley index of S. We will denote this homotopy type by $h(S)$, and the corresponding *homology Conley index* $CH_*(S)$ is defined by the reduced homology $\tilde{H}_*(N/L)$. When (N, L) is a regular index pair, this homology is more conveniently given as $H_*(N, L)$.

The most familiar example is a hyperbolic critical point. If x is a hyperbolic critical point with unstable dimension u, then x is isolated and admits an index pair of the form $(D^u \times D^s, S^{(u-1)} \times D^s)$. Clearly, $h(x) \simeq D^u/S^{(u-1)} \simeq S^u$. This homotopy class is denoted Σ^u, and the corresponding homology Conley index is

$$CH_k(x) \cong H_k(D^u, S^{(u-1)}) = \begin{cases} Z, & k = u \\ 0, & \text{otherwise.} \end{cases}$$

It is this homological structure that we assume in (**H2**) for all of the Morse sets in $\mathscr{A}$.

2.3. *Connection Matrices.* Once we have a Morse decomposition of an isolated invariant set S, we have a Conley index $CH_*(S(I))$ defined for every interval $I \subset P$. We now look at how the indices of the various $S(I)$ are interrelated, and what those relations reveal about the flow on S.

In the simplest case, an attractor-repeller decomposition, there are three isolated invariant sets, S, A and R. Associated with these is an *index triple*: a triple (N, M, L) with the property that (N, L) is an index pair for S; (M, L) is an index pair for A; and (N, M) is an index pair for R. The homology long exact sequence of the triple then defines a long exact sequence, called the *attractor-repeller sequence*,

$$\cdots \to CH_p(A) \to CH_p(S) \to CH_p(R) \xrightarrow{\partial} CH_{p-1}(A) \to \cdots$$

that relates the homology Conley indices of S, A, and R. The map ∂ is called the *connection homomorphism*. Its basic property is that if $\partial \neq 0$, then there exist connecting orbits from R to A in S. In some cases, it can give more refined information about the set of connecting orbits. For example, if A and R have the indices Σ^p and Σ^{p-1}, respectively, then the only nontrivial portion of the attractor-repeller sequence is

$$0 \to CH_{p+1}(S) \to Z \xrightarrow{\partial} Z \to CH_p(S) \to 0.$$

Then ∂ can be thought of as an integer.

If the flow has the additional property that $W^s(A)$ and $W^u(R)$ intersect transversely, then the connecting orbit set consists of a disjoint set of orbits $\gamma_1, ..., \gamma_N$. Each $R \cup \gamma_i \cup A$ is an isolated invariant set with attractor-repeller pair (A, R). The corresponding connection homomorphism $\partial_i = \pm 1$, with the sign depending on the orientation of the stable and unstable manifolds. The connection homomorphism for all of S is then $\partial = \sum_i \partial_i$. In particular, there are at least ∂ connecting orbits, and that the number of connecting orbits is equal to $\partial \bmod 2$ [9].

In general, given a Morse decomposition with an admissible order $(P, <)$, there is an attractor-repeller sequence for every adjacent pair of intervals in P. In [5], Franzosa introduced *connection matrices* as devices for simultaneously encoding the information expressed in all of these sequences. In brief, connection matrices are matrices defined on the sum of the homology indices of the Morse sets, and which, when treated as boundary maps, allow all of the attractor-repeller sequences to be reconstructed.

More precisely, for every interval $I \subset P$, let $C_* \Delta(I) = \bigoplus_{p \in I} CH_*(S(p))$. Suppose that $\Delta(P)\colon C_* \Delta(P) \to C_* \Delta(P)$ is a degree -1 endomorphism such that

(1) $\Delta(P)^2 = 0$

(2) If $p \nless q$, then $\Delta(p, q)\colon CH_*(S(q)) \to CH_*(S(p))$ is zero.

Such a matrix is said to be an *upper triangular boundary map*. Given any two intervals $I, J \subset P$, define $\Delta(I, J)\colon C_*\Delta(J) \to C_*\Delta(I)$ to be the obvious restriction of $\Delta(P)$, and denote $\Delta(I, I)$ by $\Delta(I)$. Then the two conditions on $\Delta(P)$ are inherited by $\Delta(I)$. In particular, given an adjacent pair of intervals I, J in P, there is a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & C_*\Delta(I) \arrow[r, "i"] \arrow[d, "\Delta(I)"] & C_*\Delta(IJ) \arrow[r, "p"] \arrow[d, "\Delta(IJ)"] & C_*\Delta(J) \arrow[r] \arrow[d, "\Delta(J)"] & 0 \\ 0 \arrow[r] & C_*\Delta(I) \arrow[r, "i"] & C_*\Delta(IJ) \arrow[r, "p"] & C_*\Delta(J) \arrow[r] & 0 \end{tikzcd}$$

where i and p are the inclusion and projection homomorphisms respectively. This can be interpreted as a short exact sequence of chain complexes, with the matrices Δ acting as boundary homomorphisms. If the homology of the complex $\{C_*\Delta(I), \Delta(I)\}$ is denoted $H_*\Delta(I)$, then the diagram above produces a long exact sequence

$$\to H_k\Delta(I) \xrightarrow{i_*} H_k\Delta(IJ) \xrightarrow{p_*} H_k\Delta(J) \xrightarrow{[\Delta(J, I)]} H_{k-1}(I) \to$$

So an upper triangular boundary map produces a long exact sequence for every adjacent pair of intervals. $\Delta(P)$ is a connection matrix if all of these sequences are canonically isomorphic to the attractor-repeller sequences. That is, we require that, for every interval I, there is a isomorphism $\phi(I)\colon H_*\Delta(I) \to CH_*(S(I))$ such that $\phi(p) = id$ for every $p \in P$, and for every adjacent pair of intervals I, J, there is a commutative diagram

$$\begin{tikzcd} H_k\Delta(I) \arrow[r, "i_*"] \arrow[d, "\phi(I)"] & H_k\Delta(IJ) \arrow[r, "p_*"] \arrow[d, "\phi(IJ)"] & H_k\Delta(J) \arrow[r, "{[\Delta(J, I)]}"] \arrow[d, "\phi(J)"] & H_{k-1}(I) \arrow[d, "\phi(I)"] \\ CH_k(S(I)) \arrow[r, "i_*"] & CH_k(S(IJ)) \arrow[r, "p_*"] & H_k(S(J)) \arrow[r, "\partial"] & H_{k-1}(S(I)) \end{tikzcd}$$

If p and q are adjacent elements of P, then $[\Delta(p, q)] = \Delta_{qp}$, and the isomorphisms [2] $\phi(p)\colon H_*\Delta(p) \to H_*(S_p)$, $\phi(q)\colon H_*\Delta(q) \to H_*(S_q)$ are identity maps. That is, the entry Δ_{qp} between adjacent elements in P is simply the connection homomorphism of the attractor-repeller pair (S_p, S_q). Such elements of the connection matrix are said to be *flow-defined.*

This distinction is important, because the other entries of the matrix need not be uniquely defined. In general, the remaining entries depend on the indices of the Morse sets and the partial order on P. However, the properties required for a connection matrix do put some constraints on these entries:

- If there is no k such that $CH_k(S_q)$ and $CH_{k+1}(S_p)$ are both non-zero, then $\Delta_{qp}=0$.
- If $q \leqslant p$, then $\Delta_{pq}=0$.

For example, in the presence of (**H2**), Δ_{qp} can only be non-zero if $n(q)=n(p)-1$. If we further require that p and p' are unrelated in the partial order when $n(p)=n(p')$, then a pair p, q with $n(q)=n(p)-1$ will either be adjacent or unrelated in the partial order. In either case, the entry Δ_{qp} will be uniquely determined. It follows then that there is a unique connection matrix $\Delta(P)$ for $(P, <)$.

This uniqueness of the matrix is not an end in itself, but is central to the interpretation of the connection matrix. If $<$ is an admissible partial order, then it refines the flow-defined order $<_f$, and any connection matrix for $(P, <_f)$ is a connection matrix for $(P, <)$. Thus, $\Delta(P)$ is the unique connection matrix for $<_f$. Then, if there is a chain $p_1, p_2, ..., p_k$ in P with $\Delta_{p_1 p_2}\Delta_{p_2 p_3}\cdots\Delta_{p_{k-1} p_k}\neq 0$, then $p_1 <_f p_2 <_f \cdots p_k$. That is, the algebra of the connection matrix detects connecting orbits.

Unfortunately, the algebra may not detect all connecting orbits. In Section 6, some examples will be presented to show that, even in the presence of hypothesis (**H2**), and even with a unique flow-defined connection matrix, there may be connecting orbits which are not reflected in the algebra. Our construction not only requires that the flow-defined order be known, but uses in a strong way that the algebra detects the flow-defined order (i.e. to prove surjectivity in Section 5). Consequently, we have added (**H3**) an additional hypothesis to this effect.

2.4. *Semi-conjugacies.* Another important aspect of the index will be its behavior under semi-conjugacies (cf. [10, 11]). The essence of the matter is that is that the index theory is natural with respect to semi-conjugacies, as long as one works with pre-images rather than images. A technicality is that the semi-conjugacy must be a proper map: pre-images of compact sets must be compact. That is, if $f\colon X\to Y$ is a proper semi-conjugacy, and S an isolated invariant set in Y with index pair (N, L), then $T=f^{-1}(S)$ is an isolated invariant set in X with index pair $(f^{-1}(N), f^{-1}(L))$. Thus there is an index homomorphism $f_*\colon CH_*(T)\to CH_*(S)$.

If $\{S_p\}$ is a Morse decomposition of S, then $\{T_p=f^{-1}(S_p)\}$ is a Morse decomposition of T, and any admissible ordering on S gives an admissible ordering on T. Thus we can use the same ordering for both decompositions, and if I is an interval in that ordering, there is a map $CH_*(T(I)) \to CH_*(S(I))$. Moreover, the attractor-repeller sequence is natural: if I and J are adjacent intervals with $I<J$, there is a commutative diagram

$$\begin{array}{ccccccc}
\xrightarrow{\partial_T} & CH_k(T(I)) & \longrightarrow & CH_k(T(IJ)) & \longrightarrow & CH_k(T(J)) & \longrightarrow \\
 & \Big\downarrow f_* & & \Big\downarrow f_* & & \Big\downarrow f_* & \\
\xrightarrow{\partial_S} & CH_k(S(I)) & \longrightarrow & CH_k(S(IJ)) & \longrightarrow & CH_k(S(J)) & \longrightarrow
\end{array}$$

While f_* does intertwine the connection homomorphisms, it does not in general intertwine connection matrices. That is, if $\Delta_S(P)$ is a connection matrix for S and $\Delta_T(P)$ is a connection matrix for T, we can form the diagram

$$\begin{array}{ccc}
\bigoplus_{p\in P} CH_k(T_p) & \xrightarrow{\Delta_T(P)} & \bigoplus_{p\in P} CH_{k-1}(T_p) \\
\Big\downarrow \bigoplus_{p\in P} f_{p*} & & \Big\downarrow \bigoplus_{p\in P} f_{p*} \\
\bigoplus_{p\in P} CH_k(S_p) & \xrightarrow{\Delta_S(P)} & \bigoplus_{p\in P} CH_{k-1}(S_p)
\end{array}$$

We can form the diagram, but it does not ncessarily commute—even if $\Delta_S(P)$ and $\Delta_T(P)$ are unique. One aspect of Theorem 1.2 is that the semi-conjugacy we will construct will give us a commutative diagram. In fact, it will show that the connection matrices on $\mathscr{A}$ and $\mathscr{M}$ are conjugate.

2.5. *Invariance Properties.* Since the Conley index of an isolated invariant set depends on both the topology of the ambient space and the flow itself, it is natural to ask how changes in either the space or the flow affect the index. These issues will be of direct importance in constructing the semi-conjugacy, as both changes in the ambient space and the flow will be involved.

The invariance of the index under changes in the flow, sometimes referred to as the *continuation property* or *homotopy invariance*, is one of the fundamental properties of the Conley index. It enters into the index in two ways. First, it facilitates the computation of the index. If an isolated invariant set in a flow of interest is related by continuation to an isolated invariant set in a simpler flow, then the computation in the simple flow will be valid for the complicated flow. Second, any results whose hypotheses are formulated in terms of the Conley index will be stable under perturbations of the flow.

Given a parameterized family of flows ϕ_λ on X, parameterized by a locally arcwise-connected parameter space Λ, suppose that N is an isolating neighborhood in the λ_0-flow for some $\lambda_0 \in \Lambda$. Then we have the following invariance properties:

(1) There is a neighborhood U of λ_0 in Λ such that N is an isolating neighborhood for ϕ_λ for every $\lambda \in U$. The maximal invariant sets $\mathrm{Inv}_\lambda(N)$ are said to be *related by continuation.*

(2) We extend this by transitivity: two isolated invariant sets S_0 and S_1 under the flows ϕ_0 and ϕ_1 are related by continuation if there is a sequence of paths $\omega_1, ..., \omega_k$ in Λ and a collection of compact sets N_i in X such that $\omega_i(1) = \omega_{i+1}(0)$, N_i is an isolating neighborhood for all flows $\phi_{\omega_i(t)}$; $\omega_1(0) = \lambda_0$ and $\omega_k(1) = \lambda_1$; and $S_0 = \mathrm{Inv}_{\lambda_0}(N_1)$, $S_1 = \mathrm{Inv}_{\lambda_1}(N_k)$.

(3) The Conley index is stable under continuation: If S_0 and S_1 are related by continuation, then the homotopy types $h(S_0)$ and $h(S_1)$ are equal. In particular, the homology Conley indices $CH_*(S_0)$ and $CH_*(S_1)$ are isomorphic.

This general form of homotopy invariance will be important in applying our results, but does not enter into the proofs of those results. However, a different type of modification of the flow will appear: time reparameterizations. Given a flow $\phi: X \times R \to X$, a reparameterization of ϕ is a function $\theta: X \times R \to R$ such that

- The function $\theta_x: R \to R$ defined by $\theta_x(t) = \theta(x, t)$ is an orientation-preserving homeomorphism for all $x \in X$.
- For all $x \in X$, $\theta(x, 0) = 0$.
- For all $x \in X$, $\theta(x, s+t) - \theta(x, s) = \theta(\phi(x, s), t)$

These conditions guarantees that $\phi'(x, t) = \phi(x, \theta(x, t))$ will be a well-defined flow. Clearly, the trajectories of ϕ and ϕ' coincide, and have the same orientation. It is not surprising then that they have the same index properties:

PROPOSITION 2.1. *If S is an isolated invariant set under the flow ϕ with index pair (N, L), and ϕ' is a time reparameterization of ϕ, then S is an isolated invariant set under ϕ' and (N, L) is an index pair.*

Proof. Rather than formulating this as a continuation result by constructing a family of flows containing ϕ and ϕ', it is easier to verify directly from the definition that (N, L) is an index pair for S under ϕ'. All of these follow immediately from the fact that ϕ and ϕ' have the same trajectories:

- S is still invariant; and every orbit in $N\backslash(L \cup S)$ leaves N.
- L is still positively invariant in N.
- If an orbit leaves N, it still passes first through L.

The other form of invariance we must consider involves changing the ambient space. We have assumed that $\mathscr{A}$ is an attractor in some larger system X. Of course, the case $X = \mathscr{A}$ is admissible, as long as X is compact. But if $\mathscr{A} \neq X$, then the following issue arises. The semi-conjugacy we will construct is defined only on $\mathscr{A}$, not on X. When we show that the semi-conjugacy is surjective in Section 5, we use the homology Conley index homomorphism $f_*: CH_*(S_p) \to CH_*(M_p)$. But, since f is only defined on $\mathscr{A}$, it is the index of S_p relative to $\mathscr{A}$ that appears, not the index of S_p relative to X. On the other hand, it is the index of S_p relative to X that we want in the hypotheses. The reason for this is computational. In order to make the hypotheses as easy to verify as possible, we want to assume as little as possible about $\mathscr{A}$. Obviously, requiring information about S_p sits in $\mathscr{A}$ is more demanding than requiring information about how S_p sits in X.

This distinction between the index relative to $\mathscr{A}$ and relative to X is only relevant if the two indices are different. In general, this can happen. If A is a closed invariant subset of X, and S is an isolated invariant set in X, then $S \cap A$ is isolated in A and the inclusion map $\iota: A \to X$ is a semi-conjugacy. There is then a homomorphism $\iota_*: CH_*(S \cap A) \to CH_*(S)$. Even if $S \subset A$, ι_* need not be an isomorphism and the two homology indices need not coincide.

While this is true in general, the assumption that $\mathscr{A}$ is an attractor can be exploited to avoid this difficulty when $S \subset \mathscr{A}$. To avoid over-burdening the notation, we will use $CH_*(\mathscr{A} \cap S)$ to denote the homology Conley index of S relative to $\mathscr{A}$; and $CH_*(S)$ to denote the homology Conley index of S relative to X.

THEOREM 2.2. *Let H denote a homology theory that satisfies the continuity axiom. If $\mathscr{A}$ is a compact attractor in a flow X, and $S \subset \mathscr{A}$, then S is isolated in $\mathscr{A}$ if and only if it is isolated in X, and the inclusion $\iota: \mathscr{A} \to X$ induces an isomorphism $\iota_*: CH_*(\mathscr{A} \cap S) \to CH_*(S)$.*

Proof. If (N, L) is an index pair for S in X, then $(\mathscr{A} \cap N, \mathscr{A} \cap L)$ is an index pair for S in $\mathscr{A}$. We can choose N arbitarily close to $\mathscr{A}$, so that it is arbitrarily close to $\mathscr{A} \cap N$. The continuity axiom implies that there are choices of N such that the inclusion $\iota_*: H_*(\mathscr{A} \cap N, \mathscr{A} \cap L) \to H_*(N, L)$ is an isomorphism. ∎

That is, as long as we use a homology theory that satisfies the continuity axiom, we do not need to worry about the ambient space. We will assume for the rest of this paper that H is such a homology theory.

3. THE SIMPLICIAL MODEL

The construction of the simplicial model $\mathscr{M}(P, <)$ is very natural, and the properties of the model are easy to establish. As we will see, the hypotheses on the Morse decomposition imply that all of the information needed to construct the complex is carried by the poset $(P, <)$. The first step in developing $\mathscr{M}(P, <)$ is to isolate the most important features of the partial order. While the simplicial complex can be constructed in a straightforward fashion from P, it is not enough to construct the complex itself. We must put a flow and a Lyapunov function on $\mathscr{M}(P, <)$, and lay the groundwork for constructing the semiconjugacy in the next section. To facilitate these steps, we will take a slightly more circuitous path to the construction of $\mathscr{M}(P, <)$.

3.1. *The Partial Order.* The hypotheses (**H2**) and (**H3**) put some strong restrictions on the poset $(P, <)$, which will in turn put restrictions on the complex $\mathscr{M}(P, <)$. Some of these restrictions are:

PROPOSITION 3.1. *If $p \in P$ and $C \subset P$ is a maximal totally ordered chain emanating from p (i.e., C is totally ordered with p as its maximal element, and there is no $C' \supset C$ with those properties), then C has $n(p)+1$ elements.*

Proof. If $n(p) > 0$, then S_p is not an attractor and so has a non-empty unstable set $W^u(S_p)$. Since $W^u(S_p) \subset \mathscr{A}$, every $x \in W^u(S_p)$ has $\omega(x) \subset \mathscr{A}$, and hence has $\omega(x) \subset S_q$ for some $q \in P$. That is, for every p with $n(p) > 0$, there is a $q < p$. If p and q are adjacent, then Δ_{qp} is an isomorphism and in particular $n(q) = n(p) - 1$.

Now, if $p > p_1 > p_2 > \cdots p_m$ is a maximal chain emanating from p, then every p_i and p_{i+1} are adjacent and $n(p_{i+1}) = n(p_i) - 1$. Clearly, this implies $n(p_i) = n(p) - i$. Further, the last step in the chain must have $n(p_m) = 0$, so, $m = n(p)$. ∎

One of the important features of the graph will prove to be the number of edges emanating from a vertex. To determine the situation in low dimensions (i.e., p with $n(p) = 1$), we first require the following lemma.

LEMMA 3.2. *If $n(p) = 0$ and U_p is a positively invariant isolating neighborhood of S_p in $\mathscr{A}$, then U_p is path connected and acyclic.*

Proof. If U_p is a positively invariant isolating neighborhood of S_p, then $(U_p, \varnothing)$ is an index pair for S_p, and $CH_*(S_p) = H_*(U_p)$. By hypothesis (**H2**), the reduced homology of $CH_*(S_p)$ is trivial. The reduced homology of $H_*(U_p)$ is thus also trivial, so U_p is path connected and acyclic.

PROPOSITION 3.3. *For every p with $n(p)=1$, there are two elements $q^+, q^- \in P$ with $q^\pm < p$.*

Proof. From Proposition 3.1, if $n(p)=1$, then there is at least one $n(q)=0$ with $q<p$. The set $S(A_p^+)$ is an attractor, with attractor-repeller decomposition $(\bigsqcup_{q\in A_p} S_q, S_p)$. The index triple for this decomposition is $(N_2, N_1, \emptyset)$, with N_2 a positively invariant neighborhood of $S(A_p^+)$ and N_1 a positively invariant neighborhood of $\bigsqcup_{q\in A_p} S_q$. We can, without loss of generality, take these neighborhoods to be arbitrarily close to the attractors they isolate. Moreover, the set N_1 is a disjoint union of positively invariant neighborhoods N_q of the attractors S_q. Since $(N_q, \emptyset)$ is an index pair for S_q, hypothesis (**H2**) implies that N_q is path-connected and acyclic. Consider now the attractor-repeller sequence

$$\to CH_1(S(A_p^+)) \to CH_1(S_p) \xrightarrow{\Delta_{A_p,p}} \bigoplus_{q\in A_p} CH_0(S_q) \to CH_0(S(A_p^+)) \to .$$

The known values for $CH_*(S_p)$ and $CH_*(S_q)$, and the injectivity of $\Delta_{A_p,p}$ reduce this to

$$0 \to Z \xrightarrow{\Delta_{A_p,p}} \bigoplus_{q\in A_p} Z \to H_0(N_2) \to 0.$$

It follows then that $|A_p| = \dim H_0(N_2)+1$. The result will be complete if N_2 is connected.

Now, by choosing N_2 sufficiently close to $S(A_P^+)$, we can assume that every component of $\overline{N_2\backslash N_1}$ isolates a component of S_p. There is a unique component N^* which carries the generator of $H_1(N_2, N_1)$, and a corresponding component S_p^* of S_p. By definition of A_p, every $q\in A_p$ has $\Delta_{qp}\neq 0$, and so has $C(S_p^*, S_q)$ non-empty. And, since $S(A_P^+)$ is positively invariant, every component of S_p has a connecting orbit to at least one S_q. Thus, N_2 is path-connected.

PROPOSITION 3.4. *If $q<p$ with $n(p)=n(q)+2$, then there are a nonzero even number of $r\in P$ with $q<r<p$.*

Proof. If $n(p)=n(q)+2$, then $\Delta_{qp}=0$ and q and p cannot be adjacent. If $q<p$, then there exists at least one r with $q<r<p$. Since $\Delta^2=0$, $(\Delta^2)_{qp} = \sum_{q<r<p} \Delta_{qr}\Delta_{rp}=0$. Since each $\Delta_{qr}\Delta_{rp}$ is an isomorphism from Z to Z, there must be as many r with $\Delta_{qr}\Delta_{rp}=-1$ as with $\Delta_{qr}\Delta_{rp}=1$. ∎

3.2. *Constructing the Simplicial Model.* The simplicial model $\mathscr{M}(P, <)$ can be thought of as a the geometric realization of the partial order $(P,<)$. An inductive construction of this geometric realization is:

(1) The elements of P are the 0-skeleton.

(2) Form the 1-skeleton by adding an edge from p to q if $q < p$.

(3) Inductively add the k-skeleton by filling in all possible k-simplices. That is, if all of the $k-1$ simplices required to form $\partial\sigma$ are present in the $(k-1)$-skeleton, then add σ to the k-skeleton.

A non-inductive formulation of this [4] is:

DEFINITION 3.5. $[p_0 p_1 \cdots p_n]$ is a simplex in $\mathscr{M}(P, <)$ if and only if each $p_i < p_{i+1}$. If $I \subset P$ is an interval in P, let $M(I)$ denote the maximal subcomplex spanned by the vertices in I.

That is, σ is a simplex in $\mathscr{M}(P, <)$ if and only if there is an ordering of its vertices $p_0, p_1, ..., p_n$ such that $p_0 < p_1 < \cdots < p_n$.

Our goal is to construct the simplicial complex and define a flow and a Lyapunov function on it. These constructions should have the following properties:

- Each simplex is invariant under the flow.
- If $p_0 < p_1 < \cdots < p_n$, then the open simplex $(p_0 p_1 \cdots p_n) \subset W^s(p_0) \cap W^u(p_n)$.
- The height function $h(t) = \sum_{p \in P} n(p)\, t_p$ is a Lyapunov function.

While it is clear that we can define such a flow on $\mathscr{M}(P, <)$, we must also structure the flow in such a way that we can define a semi-conjugacy from $\mathscr{A}$ to $\mathscr{M}$. It is this ultimate goal that justifies the following very unintuitive construction, derived from that of [12].

The idea is to form an n-simplex as a quotient of the cube $[0, n] \times I^{(n-1)}$, where $I = [0, 1]$ is the unit interval. The flow on $[0, n] \times I^{(n-1)}$ will have the form

$$\dot{x} = -g(x, t)$$

$$\dot{t} = 0$$

with $g(x, t)$ a non-negative chosen to produce a well-defined flow on the quotient. The Lyapunov function x will likewise produce a well-defined Lyapunov function on the quotient. The simplicial complex is formed by gluing the simplices together in the usual manner, and the flows on the simplices will form a well-defined flow on the complex.

To form Δ^n as a quotient of $[0, n] \times I^{n-1}$, we first define $l, r: [0, n] \times I^{n-1} \to \{0, ..., n\}$. If $(x, \tau_1, ..., \tau_{n-1}) \in [0, n] \times I^{n-1}$, define $l, r: [0, n] \times I^{n-1} \to \{0, ..., n\}$ by

$$l(x, \tau_1, ..., \tau_{n-1}) = \begin{cases} n & \text{if } \quad x = n \\ k & \text{if } \quad k \leqslant x, \tau_k = 1, \text{ and } \forall k < p < x, \tau_p \neq 1 \\ 0 & \text{if no such } k \text{ exists} \end{cases}$$

$$r(x, \tau_1, ..., \tau_{n-1}) = \begin{cases} 0 & \text{if } \quad x = 0 \\ k & \text{if } \quad x \leqslant k, \tau_k = 1, \text{ and } \forall x < p < k, \tau_p \neq 1 \\ n & \text{otherwise} \end{cases}$$

Define an equivalence relation on $[0, n] \times I^{n-1}$ by $(x, \tau) \sim (x', \tau')$ if $x = x'$, $l(x, \tau) = l(x', \tau')$, $r(x, \tau) = r(x', \tau')$ and $\tau_p = \tau'_p$ for every $l(x, \tau) < p < r(x, \tau)$. Let

$$\eta: [0, n] \times I^{n-1} \to Q = [0, n] \times I^{n-1}/\sim$$

be the quotient map. That is, we identify points on the face $\tau_k = 1$ which have the same value for x and the same values for some of the τ_p's (which ones are required to agree depends on x, via the functions l and r). This process (with $n = 3$) is illustrated in Fig. 1. The identification map collapses $\{x = 0\}$ and $\{x = 3\}$ each to a point, and collapses each of the lines $\{\tau_1 = 1, x = c \mid 0 < c \leqslant 1\}$ and $\{\tau_2 = 1, x = c \mid 2 \leqslant c < 3\}$ to a point.

Note that, since η only identifies subfaces to points, Q is homeomorphic to Δ^n (though η is not a homeomorphism). Now choose a homeomorphism $\lambda: Q \to \Delta^n$ which sends $[k, 0, ..., 0, \tau_k = 1, 0, ..., 0]$ to the vertex v_k and sends $\eta(\{\tau_{i_1} = \cdots = \tau_{i_n} = 0\})$ to the subface opposite $v_{i_i}, ..., v_{i_n}$ (i.e., the subface expressed in barycentric coordinates as $\{\tau_{i_1} = \cdots = \tau_{i_n} = 0\}$).

We now construct a flow on $[0, n] \times I^{n-1}$ which induces the desired flow on Δ^n. Let $s_n: \Delta^n \to [-n-1, 0]$ be defined by $s(t_0, ..., t_n) = -\sum_{i<j} t_i t_j$. Define a flow $\hat{\phi}: [0, n] \times I^{n-1} \times R \to [0, n] \times I^{n-1}$ by

$$\dot{x} = s_n \circ \lambda \circ \eta(x, \tau)$$

$$\dot{\tau} = 0$$

This will generate a well-defined flow $\phi: \Delta^n \times R \to R$ if $\eta\hat{\phi}(x, \tau, t) = \eta\hat{\phi}(x', \tau', t)$ when $\eta(x, \tau, t) = \eta(x', \tau', t)$. This will be the case if $l(\hat{\phi}(x, \tau, t)) = l(x, \tau)$ and $r(\hat{\phi}(x, \tau, t)) = r(x, \tau)$.

If $l(x, \tau) = k_0$ and $r(x, \tau) = k_1$, then $\tau_{k_0} = \tau_{k_1} = 1$ and no p between k_0 and k_1 has $\tau_p = 1$. Then, on the set $\{(x'', \tau'') \mid k_0 \leqslant x'' \leqslant k_1, \tau'' = \tau\}$, the zero set

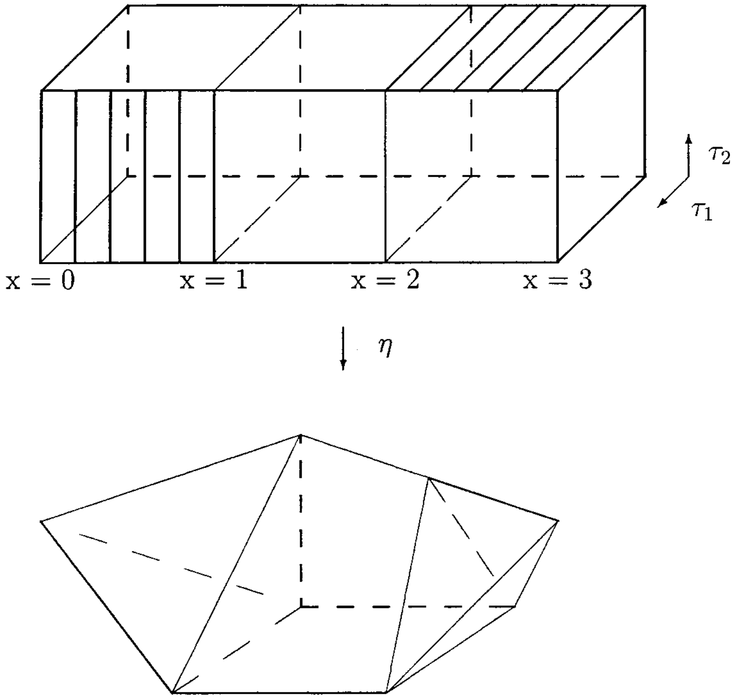

FIG. 1. The identification map η for $n = 3$.

of $s_n \circ \lambda \circ \eta$ consists of the two points (k_0, τ), (k_1, τ). Thus, these two points are the ω- and ω^*-limit points respectively of (x, τ). That is, the trajectory of (x, τ) is precisely the set

$$\{(x'', \tau'') \mid k_0 \leqslant x'' \leqslant k_1, \tau'' = \tau\},$$

and l and r are constant on that set.

The function x is a Lyapunov function on $[0, n] \times I^{n-1}$, in the sense that it is decreasing on all non-constant orbits. We can consider x as a function on Q, and define Λ: $\Delta^n \to [0, n]$ by

$$\Lambda(t_0, ..., t_n) = x(\lambda^{-1}(t_0, ..., t_n)).$$

Then Λ is a Lyapunov function on Δ^n.

The next step is to apply these constructions on simplices to define a flow and a Lyapunov function on the simplicial complex $\mathscr{M}(P, <)$. One way of viewing $\mathscr{M}(P, <)$ is that there is one maximal simplex for every maximal totally ordered chain in $(P, <)$, with the dimension of the simplex one less than the length of the chain. The simplicial complex is formed by attaching such maximal simplices to one another. To obtain a well-defined flow and Lyapunov function on $\mathcal{M}(P, <)$, we must verify that, when two maximal simplices share a common face, the flows and Lyapunov functions defined on that common face agree.

PROPOSITION 3.6. *Suppose $p_0 < \cdots < p_n$ and $q_0 < \cdots < q_m$ are two maximal chains in $(P, <)$. If $p_{i_0} = q_{i_0}, ..., p_{i_k} = q_{i_k}$, then the two flows on the k-simplex $[p_{i_0} \cdots p_{i_k}]$ agree, and the two Lyapunov functions on $[p_{i_0} \cdots p_{i_k}]$ agree.*

Proof. This is simply the observation that the flows and Lyapunov functions depend only on the dimesion and the values $n(p_i) = n(q_i)$. ▮

LEMMA 3.7. *The embedding $\hat{\imath}$: $[0, n] \times I^{n-1} \to [0, n+1] \times I^n$ defined by $\imath(x, \tau_1, ..., \tau_{n-1}) = (x, \tau_1, ..., \tau_{n-1}, 1)$ defines an embedding $\imath$: $\Delta^n \to \Delta^{n+1}$ which intertwines the flows and the Lyapunov functions. That is, there are commutative diagrams*

$$\begin{array}{ccc} \Delta^n \times R & \xrightarrow{\phi_n} & \Delta^n \\ \downarrow{\scriptstyle \imath \times id} & & \downarrow{\scriptstyle \imath} \\ \Delta^{n+1} \times R & \xrightarrow{\phi_{n+1}} & \Delta^{n+1} \end{array} \qquad \begin{array}{ccc} \Delta^n & \xrightarrow{\Lambda_n} & R \\ \downarrow{\scriptstyle \imath} & & \downarrow{\scriptstyle id} \\ \Delta^{n+1} & \xrightarrow{\Lambda_{n+1}} & R \end{array}$$

Proof. First, $\hat{\imath}$ will define an embedding if

$$(x, \tau) \sim (x', \tau') \text{ if and only if } \hat{\imath}(x, \tau) \sim \hat{\imath}(x', \tau').$$

This will be the case if $l_n(x, \tau) = l_{n+1}(x, \tau, 1)$ $r_n(x, \tau) = r_{n+1}(x, \tau, 1)$. The definitions of l_n and l_{n+1} are identical, whereas the embedding into $\tau_n = 1$ guarantees that $r_{n+1}\hat{\imath}(x, \tau) \leqslant n$. With the possibility of $r_{n+1}(x, \tau, 1) = n+1$ removed, it is easy to see that the definitions of r_n and r_{n+1} coincide.

Next, since $\imath$ maps Δ^n to the face $\{t_{n+1} = 0\}$, s_n and $s_{n+1}\imath$ agree. This implies that the flows they define coincide as well. Finally, it is a simple matter to verify that $\Lambda_n = \Lambda_{n+1}\imath$. ▮

3.3. *Index Properties of the Model Flow.* At this point, we have a model $\mathcal{M}(P, <)$, a flow ϕ: $\mathcal{M} \times R \to R$ and a function Λ: $\mathcal{M} \to R$ such that

- Each simplex is invariant under the flow.
- The vertex set is the rest point set.
- The function Λ is a Lyapunov function, with $\Lambda(p) = n(p)$.
- If $p_{i_0} < \cdots < p_{i_k}$, then the open simplex $(p_{i_0} \cdots p_{i_k}) \subset W^s(p_{i_0}) \cap W^u(p_{i_k})$.

An immediate consequence of these properties is:

PROPOSITION 3.8. *The vertex set* $\{M_p\}_{p \in P}$ *forms a Morse decomposition for the flow, with flow-defined order* $(P, <)$. *In particular, each* M_p *is an isolated invariant set.*

We need to understand the index structure of this flow. That is, we want to find isolating neighborhoods and index pairs for each M_p, then combine that information with the flow-defined order to reconstruct the connection matrix. The natural result to expect here would be that each M_p has $h(M_p) = \Sigma^{n(p)}$, and that the connection matrix on $\{M_p\}$ is conjugate to the original connection matrix on $\{S_p\}$. Somewhat surprisingly, this is not true in general. To see what is true, we use the simplicial structure of $\mathscr{M}$ to construct convenient index pairs.

PROPOSITION 3.9. *The unstable set* $W^u(M_p)$ *of* M_p *is* $M(A_p^+)\backslash M(A_p)$. *This is also the open star in* $M(A_p^+)$ *of the vertex p, and is homeomorphic to the open cone on* $M(A_p)$. *The closure of the unstable set is* $M(A_p^+)$, *which is also the closed star in* $M(A_p^+)$ *of the vertex p, and is homeomorphic to the closed cone on* $M(A_p)$.

COROLLARY 3.10. *If* Ξ_p *is a section of* $W^u(M_p)\backslash M_p$, *then* Ξ_P *is homeomorphic to* $M(A_p)$.

PROPOSITION 3.11. *There exists an index pair* (N_p, L_p) *for* M_p *such that* $(M(A_p^+), M(A_p)) \subset (N_p, L_p)$, *and the inclusion* $(M(A_p^+), M(A_p)) \to (N_p, L_p)$ *is a homotopy equivalence.*

Proof. $M(A_p^+)$ is an attractor, with basin of attraction $St(M(A_p^+))$, the set of open simplices with at least one vertex in $M(A_p^+)$. Choose N_p to be any positively invariant compact neighborhood of $M(A_p^+)$ in $St(M(A_p^+))$. Similarly, $M(A_p)$ is an attractor, with basin of attraction $St(M(A_p))$. Choose L_p to be a positively invariant compact neighborhood of $M(A_p)$ in $St(M(A_p))$. If $L_p \not\subset N_p$, replace it with $L_p \cap N_p$. It is routine to verify that (N_p, L_p) is an index pair for M_p.

Pushing forward by the flow, $(St(M(A_p^+)), St(M(A_p)))$ has a strong deformation retraction onto (N_p, L_p). But $(St(M(A_p^+)), St(M(A_p)))$ also has an obvious strong deformation retraction onto $(M(A_p^+), M(A_p))$. Composing these gives a homotopy equivalence $(M(A_p^+), M(A_p)) \to (N_p, L_p)$. ∎

COROLLARY 3.12. M_p *has the homotopy Conley index of* $\Sigma(M(A_p))$, *the suspension of* $M(A_p)$.

COROLLARY 3.13. $CH_k(M_p) \cong \tilde{H}_{k-1}(M(A_p))$.

The natural expectation is that M_p will have the homology Conley index of $\Sigma^{n(p)}$. This will be the case when $M(A_p)$ is a homology $(n(p)-1)$-sphere. We will be particularly interested in the case when Ξ_p, hence $M(A_p)$, is homeomorphic to an $(n(p)-1)$-sphere. The following example will show that, in general, $M(A_p)$ need not be a sphere, nor even a homology sphere. It will be necessary to add this as an explicit hypothesis.

EXAMPLE 3.14. Consider the flow on the 2-torus generated by the following flow on the unit square shown in Fig. 2. There are eight Morse sets, each a hyperbolic fixed point, with indices and partial order as illustrated in Fig. 3. If p is one of the points of index 2, then $M(A_p)$ is the 1-complex shown in Fig. 4. This is a wedge of cirles, but not a 1-sphere.

Since we will need to add (**H4**) as an extra hypothesis, we would like to formulate conditions in terms of the partially ordered set $(P, <)$ that imply (**H4**).

THEOREM 3.15. *Let $(P, <)$ be a partially ordered set, and $\mathscr{M}(P, <)$ the corresponding complex.*

(1) *If for every p with $n(p) > 0$ there are exactly two $r_+, r_- < p$ with $n(r_\pm) = n(p) - 1$, then every $M(A_p) \cong S^{n(p)-1}$.*

(2) *If every $M(A_p) \cong S^{n(p)-1}$, then whenever $q < p$ with $n(p) = n(q) + 2$, there are exactly two r_+, r_- with $q < r_+, r_- < p$.*

Proof. Statement (1) is proved by induction on $n(p)$. Lemma 3.3 shows that the result is always true for $n(p) = 1$. Now, if every $p \in P$ has exactly two elements of P immediately below it in the partial order, then A_p consists $2n(p)$ elements, with exactly two of each index from 0 to $n(p) - 1$.

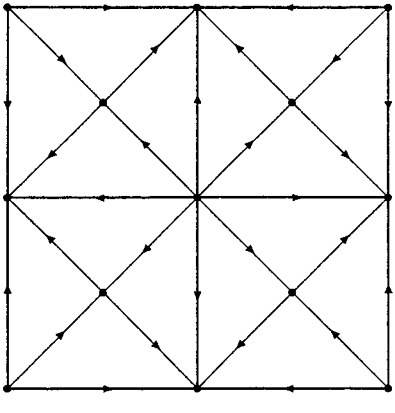

FIG. 2. Flow on the torus.

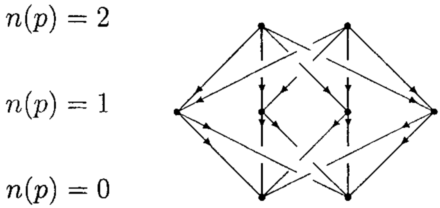

FIG. 3. Partial order of torus flow.

These have $q < r$ if and only if $n(q) < n(r)$. Figure 5 illustrates this, structure for $n(p) = 3$.

To see that this claim is true, proceed by induction on $n(p) - k$. For $k = 1$, this is true by hypothesis. If these two are r_0, r_1, then there are $q_{00}, q_{01}, q_{10}, q_{11}$ with $n(q_{ij}) = n(p) - 2$ and $q_{ij} < r_i$. By Lemma 3.4, each of the points q_{ij} has an even number of points in P that lie between it and p. Those points can only be r_0 and r_1. This implies $q_{0j} = q_{1j}$, which we now denote as q_j. Thus we have $q_0, q_1 < r_0, r_1 < p$. Now apply this argument iteratively down to the zero level.

Now, it is clear that $M(A_p) = M(A^+_{r_0}) \cup M(A^+_{r_1})$, with $M(A^+_{r_0}) \cap M(A^+_{r_1}) = M(A_{r_0}) = M(A_{r_1}) \cong S^{n(p)-2}$. Since each $M(A_{r_i})$ is an $n(p) - 2$ sphere, each $M(A^+_{r_i})$ is an $n(p) - 1$ disk. That is, $M(A_p)$ consists of two $n(p) - 1$ disks joined along a common $n(p) - 2$ sphere, and so is an $n(p) - 1$ sphere.

To prove statement (2), take any $q < p$ with $n(q) = n(p) - 2$, and take any maximal totally ordered chain $q_0 < \cdots < q_{n(q)-1} < q$ descending from q. Let σ be the corresponding $n(q)$-simplex in $M(A_p)$. If $q < r_1, \ldots, r_k < p$, then σ is a face in k simplices $[q_0 \cdots q r_i]$ in $M(A_p)$. But, since $M(A_p)$ is an $n(p) - 2$ sphere, an $n(p) - 2$ simplex can be the face of only two $n(p) - 1$ simplices. ∎

Example 3.14 shows that some hypothesis is required to ensure that each $M(A_p)$ is an $n(p) - 1$ sphere, while Theorem 3.15 gives a necessary condition and a sufficient condition for this. The following examples show that neither condition exactly characterizes the sphere condition.

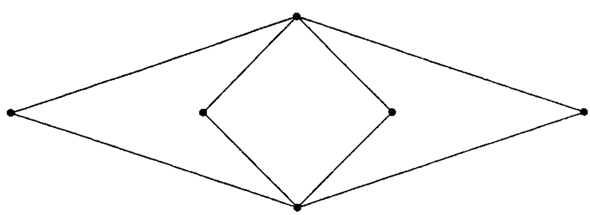

FIG. 4. $M(A_p)$ for the repelling points in the torus flow.

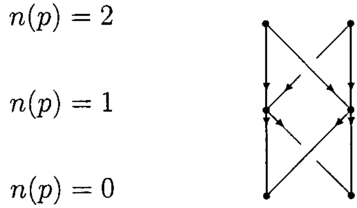

FIG. 5. Partial order of Theorem 3.15.1.

Consider the partially ordered sets P_1, P_2 of Fig. 6. In P_1, the one maximal element p_1 is adjacent to three elements, not two, but it still has a 1-sphere for $M(A_{p_1})$. On the other hand, in P_2, between the maximal element M_{p_2} and each minimal element are exactly two elements. But $M(A_{p_2})$ consists of two 1-spheres, not one.

We have seen that some hypothesis on the partial order is required to obtain the appropriate indices for the Morse sets in the model flow. The content of Theorem 1.2 is that this hypothesis will suffice for all of the additional structures we require: it will imply that the semi-conjugacy is surjective; that it induces an isomorphism on the homology indices; and that it conjugates the connection matrices on $\mathscr{A}$ and $\mathscr{M}$. As a first step towards establishing this, we have the following result.

PROPOSITION 3.16. *If each $M(A_p) \cong S^{n(p)-1}$, then the Morse decomposition $\{M_p\}_{p \in P}$ has a unique connection matrix $\Delta_M(P)$. The only nonzero entries in this matrix are the flow-defined entries ∂_{qp} for adjacent entries $q < p$. These nonzero entries are all isomorphisms.*

Proof. By construction, the qp entry of $\Delta_M(P)$ must be zero unless $n(q) = n(p) - 1$ and $q < p$. That corresponds exactly to q and p being adjacent in $(P, <)$, so the only non-zero entries are the flow-defined connection maps. To compute the connection map Δ_{qp}, we must consider the

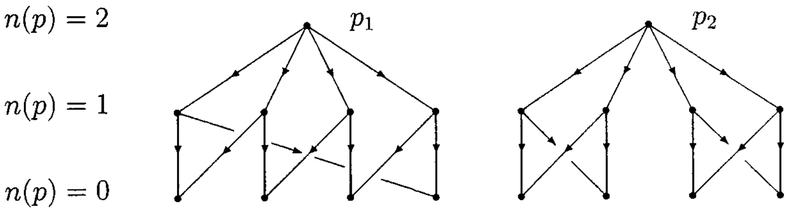

FIG. 6. Counter-example partial orders.

attractor-repeller sequence of $(M(qp); M_q, M_p)$. The relevant portion of the sequence is

$$CH_{n(p)}(M(qp)) \to CH_{n(p)}(M_p) \xrightarrow{\partial_{qp}} CH_{n(q)}(M_q) \to CH_{n(q)}(M(qp)).$$

The homomorphism ∂_{qp} will be an isomorphism if and only if $CH_*(M(qp)) = 0$.

Let $Q \subset P$ consist of $A_p \backslash \{q\}$. Then $(M(A_p^+), M(Q))$ is (up to a strong deformation retraction) an index pair for $M(qp)$. As the cone on $M(A_p)$, $M(A_p^+)$ has trivial homology. Thus $CH_*(M(qp))$ will be trivial if $H_*(M(Q))$ is. Since $A_p = \{q\} \cup Q$, the $n(q)$ sphere $M(A_p) = M(A_q^+) \cup M(Q)$, with $M(A_q^+) \cap M(Q) = M(A_q) \cong S^{n(q)}$. That is, $M(Q) = M(A_p) \backslash (M(A_q^+) \backslash M(A_q))$. Since $M(A_q)$ is a sphere of dimension $n(q) - 1$, $M(A_q^+) \backslash M(A_q)$ is an open $n(q)$-disk, and $M(Q)$ is the complement of an open $n(q)$-disk in an $n(q)$-sphere. Clearly, $M(q)$ is a closed $n(q)$-disk, and so is contractible. ∎

This proposition asserts that entries of $\Delta_M(P)$ are either isomorphisms or zeroes, and are isomorphisms if and only if the corresponding entries in the original matrix $\Delta(P)$ on $\mathscr{A}$ are. However, it stops just short of asserting that $\Delta_M(P)$ and $\Delta(P)$ are conjugate. In fact, we will see in Section 5 that one of the consequences of the construction of the semi-conjugacy f will be that it conjugates $\Delta_M(P)$ to $\Delta(P)$. The upshot of this is that, when each $M(A_p)$ is an $n(p) - 1$ sphere, the Morse decomposition $\{M_p\}$ completely reproduces the homological information of the Morse decomposition $\{S_p\}$.

4. THE SEMI-CONJUGACY

Having constructed the complex $\mathscr{M}(P, <)$ and its dynamics, we are now ready to define the semi-conjugacy $f\colon \mathscr{A} \to \mathscr{M}(P, <)$. There are two basic ingredients to the construction of f. First, we choose neighborhoods in $\mathscr{A}$ about the Morse sets S_p, and define *transit time* functions τ_p that measure the time an orbit spends in each of these neighborhoods. Next, we construct a Lyapunov function Λ on $\mathscr{A}$ that is compatible with these transit time functions. Intuitively, the semi-conjugacy is constructed from these functions in the following steps:

(1) An orbit $x \cdot R$ is mapped into the simplex spanned by the points $p \in P$ with $\tau_P(x) \neq 0$.

(2) Two of the τ_p functions will be infinite; the others will be finite. The finite-valued transit time functions and the Lyapunov function give coordinates that define the image of x in the simplex.

One technicality in this will be reparameterizing the flow on $\mathscr{A}$ to obtain the needed compatibility between the Lyapunov function and the transit time functions.

In this section, we will find it convenient to consider infinite-valued functions. This will have the obvious interpretation: we will consider $[-\infty, \infty]$ to be the two-point compactification of R, and will use $\tan^{-1}$ as the canonical homeomorphism to $[-1, 1]$.

4.1. *The Transit Time Functions.* To construct the transit time functions, we begin by making use of the connection between attractor-repeller pairs and Lyapunov functions. For each n, let $P_n = \{p \in P \mid n(p) \leqslant n\}$. Then P_n is an attracting interval, with a corresponding attractor $\mathscr{A}_n$ in $\mathscr{A}$. Let $\mathscr{A}_n^*$ be its dual repeller.

LEMMA 4.1. *There exists a disjoint collection of sets $\{L_n\}_{n \geqslant 0}$ such that*

(1) *Each L_n is a cross-section to the flow: if $x \in L_n$, then $x \cdot t \notin L_n$ for any $t \neq 0$.*

(2) *$x \cdot R \cap L_n \neq \varnothing$ if and only if $x \in C(S_p, S_q)$ for some $p, q \in P$ with $n(p) > n \geqslant n(q)$.*

(3) *The function $\rho_n \colon \mathscr{A} \to [-\infty, \infty]$ defined by*

$$\rho_n(x) = \begin{cases} t, & x \cdot t \in L_n \\ \infty, & x \cdot R \cap L_n = \varnothing,\ n(\omega(x)) > n \\ -\infty, & x \cdot R \cap L_n = \varnothing,\ n(\omega^*(x)) < n \end{cases}$$

is continuous.

(4) *For every $n < m$ and every $x \in \mathscr{A}$, $\rho_n(x) \geqslant \rho_m(x)$, with $\rho_n(x) > \rho_m(x)$ if either value is finite.*

Proof. Corresponding to each attractor-repeller pair $(\mathscr{A}_n, \mathscr{A}_n^*)$ is a Lyapunov function $\zeta_n \colon \mathscr{A} \to [0, 1]$ with $\mathscr{A}_n = \zeta^{-1}(0)$, $\mathscr{A}_n^* = \zeta_n^{-1}(1)$ and ζ_n strictly decreasing on orbits in $C(\mathscr{A}_n^*, \mathscr{A}_n)$ [21]. Choose values $0 < t_1 < t_2 < \cdots < 1$ such that, if $i < j$, then $\zeta_i^{-1}(t_i) \subset \zeta_j^{-1}([0, t_j))$. Then let $L_n = \zeta_n^{-1}(t_n)$. All of the required properties follow in a routine fashion. ∎

The function $\rho_n(x)$ is the *arrival time* at L_n. These functions are clearly non-increasing on orbits: monotone-decreasing for orbits that intersect L_n; constant (at $\pm\infty$) for orbits that do not.

Let $Q_n = \{x \in \mathscr{A} \mid \rho_n(x) \leqslant 0\}$ and $T_n = \overline{Q_n \backslash Q_{n-1}}$. T_n is an isolating neighborhood for $S(P_n)$, the union of Morse sets with index $n(p) = n$. For each n, there is a positive minimum transit time $2\mu_n = \min_{x \in T_n} \{\rho_{n-1}(x) - \rho_n(x)\} > 0$ that orbits require to pass through T_n.

We want to measure the time orbits in $\mathscr{A}$ spend near each Morse set. Since orbits in $W^u(S_p)$ and $W^s(S_p)$ spend infintely long amounts of time near S_p, we will need to allow infinite values for these functions. The following construction allows us to define these functions in a simple fashion.

Lemma 4.2. *There exists a collection of compact subsets $\{N_p\}_{p\in P}$ of $\mathscr{A}$ such that*

(1) *N_p is an isolating neighborhood for S_p.*

(2) *If $C(S_p, S_q)=\varnothing$, then $N_p\cdot R\cap N_q=\varnothing$.*

(3) *There exist continuous functions $\tau_p^+, \tau_p^-: N_p\cdot R\to[-\infty, \infty]$ such that, for all $x\in N_p\cdot R$, $\tau_p^-(x)\leqslant\tau_p^+(x)$ and $x\cdot R\cap N_p=x\cdot[\tau_p^-(x), \tau_p^+(x)]$.*

(4) *There are constants $v_n>\mu_n$ such that, if $n(p)=n$, then $\tau_p^+|_{N_p\cdot R\cap L_{n-1}}=-v_n$ and $\tau_p^-|_{N_p\cdot R\cap L_n}=v_n$.*

(5) *The function $\tau_p: T_n\cdot R\to[0, \infty]$ defined by*

$$\tau_p(x)=\begin{cases}\tau_p^+(x)-\tau_p^-(x), & x\cdot R\cap N_p\neq\varnothing\\ 0, & x\cdot R\cap N_p=\varnothing\end{cases}$$

is continuous.

Proof. For every attracting interval $A\subset P$, the corresponding attractor-repeller pair $(\mathscr{A}(A), \mathscr{A}^*(A))$ admits a Lyapunov function $\zeta_A: \mathscr{A}\to[0, 1]$. For each $p\in P$, let

$$\tilde{N}_p=\left(\bigcap_{p\in A}\zeta_A^{-1}([0, \varepsilon])\right)\cap\left(\bigcap_{p\notin A}\zeta_A^{-1}([1-\varepsilon, 1])\right).$$

We can choose ε sufficiently small that $\tilde{N}_p\subset\operatorname{int}(T_n)$. Then $\tilde{N}_p$ is an isolating neighborhood for S_p, and if $p, q\in P$ are unrelated in the partial order on P, then $\tilde{N}_p\cdot R\cap\tilde{N}_q=\varnothing$.

Now, since $\rho_{n-1}-\rho_n$ is continuous on T_n and infinite on the stable and unstable sets of the Morse sets, there is a $v_n>\mu_n$ such that

$$(\rho_{n-1}-\rho_n)^{-1}([2v_n, \infty])\subset\bigcup_{p\in P_n}\tilde{N}_p\cdot R.$$

Let

$$N_p=\rho_{n-1}^{-1}([v_n, \infty])\bigcap\rho_n^{-1}([-\infty, -v_n])\bigcap\tilde{N}_p.$$

On $N_p \cdot R \subset \tilde{N}_p \cdot R$, the functions τ_p^+, τ_p^- are then given by $\tau_p^+(x) = \rho_{n-1}(x) - v_n$ and $\tau_p^-(x) = \rho_n(x) + v_n$. These are clearly continuous, with constant values $-v_n$ and v_n on L_{n-1} and L_n, respectively.

On $\tilde{N}_p \cdot R$, $\tau_p = \max\{\rho_{n-1} - \rho_n - 2v_n, 0\}$ is clearly continuous. On its complement in $T_n \cdot R$, $\tau_p \equiv 0$. The choice of v_n guarantees that $\tau_p = 0$ on the boundary of $\tilde{N}_p \cdot R$ in $T_n \cdot R$, so τ_p is continuous on $T_n \cdot R$. ∎

The function τ_p is the *transit time function* through N_p. Obviously, each τ_p is constant on orbits that pass through T_n. For each orbit $x \cdot R$, let $P_x = \{p \in P \mid \tau_p(x) \neq 0\}$. Implicit in this definition is that $\tau_p(x)$ is defined for all $p \in P_x$. The properties of Lemma 4.2 guarantee that each P_x is a totally ordered subset of P. If $m < M$ are the minimal and maximal elements of P_x, then $\tau_m(x) = \tau_M(x) = \infty$ and $x \in C(S_M, S_m)$. All other $p \in T_x$ have $\tau_p(x)$ finite.

We can use the τ and ρ functions to construct two more collections of sections to the flow, and two more corresponding sets of arrival time functions. Let

$$K_n = \{x \in T_n \mid -\rho_n(x) = \rho_{n-1}(x) \leqslant v_n\}$$

and

$$K_n^- = K_n \cup \left\{ x \in \bigcup_{p \in P_n} N_p \mid \rho_n(x) = -v_n \right\}$$

$$K_n^+ = K_n \cup \left\{ x \in \bigcup_{p \in P_n} N_p \mid \rho_{n-1}(x) = v_n \right\}$$

The arrival time functions $\kappa_n^\pm : \mathscr{A} \to [0, \infty]$ for $K_n^\pm$ are

$$\kappa_n^+(x) = \begin{cases} \rho_{n-1}(x) - v_n, & x \in N_p \cdot R \ \exists p \in P_n \\ \dfrac{\rho_{n-1}(x) + \rho_n(x)}{2}, & \text{otherwise} \end{cases}$$

$$\kappa_n^-(x) = \begin{cases} \rho_n(x) + v_n, & x \in N_p \cdot R \ \exists p \in P_n \\ \dfrac{\rho_{n-1}(x) + \rho_n(x)}{2}, & \text{otherwise.} \end{cases}$$

It is easy to check that these are continuous.

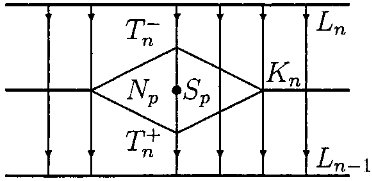

FIG. 7. The set T_n and its decomposition.

If the various isolating neighborhoods N_p are deleted from T_n, the complement is partitioned by the sections $K_n^{\pm}$ into two pieces:

$$T_n^+ = \{x \in T_n \mid \kappa_n^+(x) \leqslant 0\}$$

$$T_n^- = \{x \in T_n \mid \kappa_n^-(x) \geqslant 0\}$$

A schematic of the sets constructed is given in Fig. 7.

4.2. *Reparameterizing the flow.* We will use the transit time and arrival time functions to define a Lyapunov function $\Lambda\colon \mathscr{A} \to R$, and then to construct the semi-conjugacy $f\colon \mathscr{A} \to \mathscr{M}(P, <)$. While there are quite general existence theorems for Lyapunov functions [2, 21, 23], the construction of f will require a compatibility condition between Λ and the transit time functions τ_p: if $\Lambda(x) = \Lambda(y)$ and $\tau_p(x) = \tau_p(y)$ for every $p \in P$, then $\Lambda(x \cdot t) = \Lambda(y \cdot t)$ for all $t \in R$.

This is clearly not possible without modification of the flow. That is, we have no *a priori* control over the rate of decrease of $\Lambda(x \cdot t)$, so we simply can't guarantee that Λ will decrease at a uniform rate on each level set of the transit time functions. We therefore reparameterize the flow off of the sets N_p. We will perform the reparameterization on each $T_n^{\pm}$ separately, then piece the results together. Note that for each n, the transit times $\rho_{n-1} - \kappa_n^+$ and $\kappa_n^- - \rho_n$ through T_n^+ and T_n^- are bounded between μ_n and v_n.

If $x \in C(S_p, S_q)$, then there is a finite sequence of times $t_1 < \cdots < t_N$ such that each $x \cdot [t_i, t_{i+1}]$ lies completely in some $T_n^{\pm}$ or N_r. Define $\zeta\colon \mathscr{A} \times R \to \mathscr{A} \times R$ by taking $\zeta_1(x, t) = x$ and $\zeta_2(x, \cdot)\colon R \to R$ to be the piecewise linear, monotone increasing, function such that $\zeta(x, 0) = 0$ and

$$\zeta_2^{-1}(x, t_i) - \zeta_2^{-1}(x, t_{i-1}) = \begin{cases} \frac{1}{2}, & x \cdot [t_{i-1}, t_i] \subset T_n^{\pm} \\ \tau_r(x), & x \cdot [t_{i-1}, t_i] \subset N_r. \end{cases}$$

This defines $\zeta_2(x, \cdot)$ on a compact interval. Outside of that interval, extend ζ_2 by linearity with slope 1.

The flow is reparameterized by defining $\phi'(x, t) = \phi \circ \zeta(x, t)$.

PROPOSITION 4.3. *The function* $\phi'\colon \mathscr{A}\times R\to\mathscr{A}$ *defined by the composition* $\mathscr{A}\times R\xrightarrow{\zeta}\mathscr{A}\times R\xrightarrow{\phi}\mathscr{A}$ *is a continuous flow. The oriented orbits of* ϕ' *coincide with those of* ϕ: *for every* $x\in\mathscr{A}$, $\phi'(\{x\}\times R)=\phi(\{x\}\times R)$ *and* $\phi'(\{x\}\times R^+)=\phi(\{x\}\times R^+)$.

Proof. ϕ' will be continuous if and only if ζ_2 is. If $x\in C(S_p, S_q)$ and $\{x_n\}$ is a sequence that converges to x, then without loss of generality, each $x_i\in C(S_{p_i}, S_{q_i})$ with $q_i\leqslant q\leqslant p\leqslant p_i$. Further, since there are only finitely many ways of choosing q_i and p_i, we may restrict to a subsequence $\{x_j\}\subset C(S_{p^*}, S_{q^*})$ with $q^*\leqslant q\leqslant p\leqslant p^*$.

It will simplify the argument if we assume that the times t_j for x and t_{ij} for x_i have the following pattern: if the orbit passes from L_n to L_{n-1}, then there are times $t_{j-1}<t_j^-\leqslant t_j^+\leqslant t_{j+1}$ such that

$$
\begin{gathered}
x_i\cdot t_{j-1}\in L_n,\\
x_i\cdot[t_{j-1}, t_j^-]\subset T_n^-,\\
x_i\cdot[t_j^-, t_j^+]\subset T_n\backslash(\operatorname{int} T_n^+\cup\operatorname{int} T_n^-),\\
x_i\cdot[t_j^+, t_{j+1}]\subset T_n^+\\
x_i\cdot t_{j+1}\in L_{n-1}.
\end{gathered}
$$

That is, either $t_j^-=t_j^+$ and $\phi(x, t_j^\pm)\in K_n$, or $t_j^-<t_j^+$ and $\phi(x,[t_j^-, t_j^+])\subset N_r$ for some $r\in P_n$.

If $\phi(x,(-\infty, t^-])\subset N_p$ and $\phi(x,[t^+,\infty))\subset N_q$, then there are $t_i^\pm\to t^\pm$ such that $\phi(x_i, t_i^-)\in K_p^+$ and $\phi(x_i, t_i^+)\in K_q^-$. The continuity of the arrival time functions guarantees that the interval $[t^-, t^+]$ and the intervals $[t_i^-, t_i^+]$ have partitions $t^-<t_1<\cdots<t^+$, $t_i^-<t_{i1}<\cdots<t_i^+$ with the same number of partition points, and with each $t_{ij}\to t_j$. Further, on each $[t_{ij}, t_{i(j+1)}]$, the slope of $\zeta_2(x_i,\cdot)$ converges to the slope of $\zeta_2(x,\cdot)$ on $[t_j, t_{j+1}]$.

This gives continuity at (x, t) for $t\in[t^-, t^+]$. For $t>t^+$, $\phi(x,t)\in N_p$. If $x_n\to x$ and $t_n\to t$, then for n sufficiently large, there are values $t_n^+\to t^+$ such that $\phi(x_n,[t_n^+, t_n])\subset N_p$. But then, since ζ_2 has slope 1 for orbit segments inside N_p,

$$
\begin{gathered}
\zeta_2(x_n, t_n)=\zeta_2(x_n, t_n^+)+t_n-t_n^+\\
\zeta_2(x, t)=\zeta_2(x, t^+)+t-t^+
\end{gathered}
$$

The convergence of ζ_2 at (x, t^+) thus implies convergence for all $t>t^+$ (and similarly for all $t<t^-$).

To verify that ϕ' is a flow, we must relate $\zeta_2(x, \cdot)$ and $\zeta_2(x \cdot t, \cdot)$. If $x' = x \cdot t^*$, then $x' \cdot R$ passes through the same set of U's and N's, with times $t'_i = t_i - t^*$. Thus

$$\zeta_2(x', t) = \zeta_2(x, t + \zeta_2^{-1}(x, t^*)) - t^*.$$

Using this, we can compute

$$\begin{aligned}\phi'(\phi'(x, s), t) &= \phi\zeta(\phi\zeta(x, s), t)\\ &= \phi\zeta(\phi(x, \zeta_2(x, s)), t)\\ &= \phi(\phi(x, \zeta_2(x, s)), \zeta_2(\phi(x, \zeta_2(x, s)), t))\\ &= \phi(\phi(x, \zeta_2(x, s)), \zeta_2(x, s + t) - \zeta_2(x, s))\\ &= \phi(x, \zeta_2(x, s + t))\\ &= \phi'(x, s + t)\end{aligned}$$

Thus ϕ' is a flow.

Since ϕ' differs from ϕ only by a monotone-increasing time reparameterization, the coincidence statements are clear.

Let τ'_p and ρ'_n denote the transit time and arrival time functions with respect to the new ϕ' flow. The flow ϕ' was constructed in such a way that the transit times through the neighborhoods N_p were unchanged: $\tau'_p(x) = \tau_p(x)$ for all x and p. However, the ρ_n arrival time functions have been changed to give the flow the following compatibility condition:

PROPOSITION 4.4. *Suppose $x, y \in \mathscr{A}$ have $P_x = P_y$, with $\tau'_p(x) = \tau'_p(y)$ for all $p \in P_x$. If $\rho'_n(x) = \rho'_n(y)$ for some n, with $|\rho'_n(x)| < \infty$, then $\rho'_m(x) = \rho'_m(y)$ and $\kappa_m^{\pm}(x) = \kappa_m^{\pm}(y)$ for all $m \geqslant 0$.*

Proof. Without loss of generality, we can take $x, y \in L_n$. If $\tau'_p(x) = \tau'_p(y)$ for all $p \in P$, then the trajectories of x and y pass through the same collection of $T_m^{\pm}$'s and N_r's. There are then two sequences of times $t_{-k}(x) < \cdots < t_0(x) = 0 < \cdots < t_l(x)$, $t_{-k}(y) < \cdots < t_0(y) = 0 < \cdots < t_l(y)$ such that $\phi(x, [t_i(x), t_{i+1}(x)])$ and $\phi(x, [t_i(y), t_{i+1}(y)])$ lie in the same $T_m^{\pm}$ or N_r. Since $\phi'(x, \zeta_2^{-1}(x, t)) = \phi(x, t)$, the values $\zeta_2^{-1}(x, t_i(x))$, $\zeta_2^{-1}(y, t_i(y))$ give the various arrival times ρ'_m, τ'^{+}_r, τ'^{+}_r.

The time required to move under ϕ' from L_n to L_{n+1}, $\rho'_{n-1}(x) - \rho'_n(x)$, is the time required to flow from L_n to K_{n-1}^- to K_{n-1}^+ to L_{n-1}. From the definition of ζ_2^{-1}, this is $1 + \max_{r \in P_{n-1}}\{\tau_r(x)\}$. Thus $\rho'_{n-1}(x) - \rho'_n(x) = \rho'_{n-1}(y) - \rho'_n(y)$. Clearly, iterating this procedure gives $\rho'_m(x) = \rho'_m(y)$ for all m.

Since the functions $\kappa_m'^{\pm}$ have been constructed so that $\kappa_m'^{+} = \rho'_{m-1} - \frac{1}{2}$ and $\kappa_m'^{-} = \rho'_m + \frac{1}{2}$, equality of $\rho'_m(x)$ and $\rho'_m(y)$ clearly implies equality of $\kappa_m'^{\pm}(x)$ and $\kappa_m'^{\pm}(y)$.

We can now use the transit time and arrival time functions to construct a Lyapunov function $\Lambda: \mathscr{A} \to R$. Two aspects of the function are easy to define: we want $\Lambda(S_p) = n(p)$ and $\Lambda(L_n) = n + \frac{1}{2}$. The arrival time functions are used to extend these definitions to all of $\mathscr{A}$. We again work piecewise on the sets $T_n^{\pm}$ and N_r. On T_n, define

$$\Lambda(x) = \begin{cases} n + \frac{1}{2}(1 + \rho'_n(x)(1 + e^{\kappa_n'^{-}(x)} e^{\kappa_n'^{+}(x)})), & x \in T_n^{-} \\ n + \frac{1}{4}(e^{\kappa_n'^{-}(x)} - e^{\kappa_n'^{+}(x)}) & x \in \bigcup\limits_{p \in P_n} N_p \\ n + \frac{1}{2}(-1 + \rho'_{n-1}(x)(1 + e^{\kappa_n'^{-}(x)} e^{\kappa_n'^{+}(x)})) & x \in T_n^{+} \end{cases}$$

On the boundary between N_p and T_n^{+}, $\kappa_n'^{+} = 0$ and $\rho_{n-1} = \frac{1}{2}$, so the two definitions both reduce to $\Lambda(x) = n + \frac{1}{4}(e^{\kappa_n'^{-}(x)} - 1)$. Similarly, the two definitions along $N_p \cap T_n^{-}$ coincide as $\Lambda(x) = n - \frac{1}{4}(e^{\kappa_n'^{+}(x)} - 1)$, and the two definitions along $T_n^{+} \cap T_n^{-}$ (where $\kappa_n'^{+} = \kappa_n'^{-} = 0$ and $\rho_{n-1} = -\rho_n = \frac{1}{2}$) coincide as $\Lambda(x) = n$.

The significance of the reparameterization and the compatibility condition is that

THEOREM 4.5. *$\Lambda: \mathscr{A} \to R$ is a continuous Lyapunov function. If $x, y \in \mathscr{A}$ have $P_x = P_y$, $\tau_p(x) = \tau_p(y)$ for all $p \in P_x$ and $\Lambda(x) = \Lambda(y)$, then $\Lambda(\phi'(x, t)) = \Lambda(\phi'(y, t))$ for all $t \in R$.*

Proof. The continuity is clear, since Λ is continuous on each T_n and constant on each L_n. Λ will be a Lyapunov function if it is constant on each S_p and monotone decreasing on all other orbits. If $x \in S_p$ with $n(p) = n$, then $x \in N_p$ and $\kappa_n'^{-}(x) = -\infty$, $\kappa_n'^{+}(x) = \infty$. Then $e^{\kappa_n'^{-}(x)} = e^{\kappa_n'^{+}(x)} = 0$ and $\Lambda(x) = n$. For any other point in N_p, the functions $\kappa_n'^{-}(x)$ and $-\kappa_n'^{+}(x)$ are either constant at $-\infty$ or monotone decreasing along the orbit of x, with at least one of the two decreasing. The exponentials of decreasing functions are decreasing, so Λ decreases on $N_p \backslash S_p$. On the rest of T_n, the combination $e^{\kappa_n'^{-}(x)} e^{\kappa_n'^{+}(x)}$ is constant along orbits, as $\kappa_n'^{-}(x) - \kappa_n'^{+}(x)$ is just the negative of the transit time through $\bigcup_{p \in P_n} N_p$. The time dependence along orbits is carried by $\rho'_{n-1}(x)$ and $\rho'_n(x)$, which are strictly decreasing along orbits.

Since $\Lambda|_{T_n}$ takes values in $[n-\frac{1}{2}, n+\frac{1}{2}]$, $\Lambda(x)=\Lambda(y)$ only if x, y both lie in a common T_n. Since Λ is constant on the Morse sets, we can assume without loss that $x, y \notin S_p$. We can further assume without loss that $\rho'_n(x), \rho'_n(y) > -\infty$. Since the definition of Λ formulated in terms of the functions ρ'_m and $\kappa'^{\pm}_m$, Proposition 4.4 implies that $\Lambda(\phi'(x, t)) = \Lambda(\phi'(y, t))$ for all $t \in R$ if $\rho_n(x) = \rho_n(y)$.

There are two cases to be considered: $x, y \in T_n^-$ or $x \in N_p \cup T_n^-$, $y \in N_p$. In both cases, we have

$$\kappa'^+_n(x) - \kappa'^-_n(x) = \tau_p(x) = \tau_p(y) = \kappa'^+_n(y) - \kappa'^-_n(y).$$

In the first case, $\Lambda(x) = \Lambda(y)$ becomes

$$n + \tfrac{1}{2}(1 + \rho'_n(x)(1 - e^{-\tau_p(x)})) = n + \tfrac{1}{2}(1 + \rho'_n(y)(1 - e^{-\tau_p(y)}))$$

which immediately implies $\rho'_n(x) = \rho'_n(y)$. If $y \in N_p$, let $x' = \phi'(x, \kappa'^+_n(x) - \kappa'^+_n(y))$. Then $x' \in N_p$ with $\kappa'^{\pm}_n(x') = \kappa'^{\pm}_n(y)$, so $\Lambda(x') = \Lambda(y) = \Lambda(x)$. But Λ is strictly decreasing along the orbit of x, so $x' = x$ and $\kappa'^-_n(x) = \kappa'^-_n(y)$. Since $\rho'_n(x) = \kappa'^-_n(x) - \frac{1}{2}$, this implies that $\rho'_n(x) = \rho'_n(y)$.

We can summarize these results by forming the quotient space $\mathcal{Q} = \mathcal{A}/\sim$, where $x \sim y$ if $\Lambda(x) = \Lambda(y)$ and all $\tau_p(x) = \tau_p(y)$. The reparameterized flow ϕ' on $\mathcal{A}$ defines a flow $\tilde{\phi}$ on $\mathcal{Q}$, and Λ defines a Lyapunov function $\tilde{\Lambda}$ on $\mathcal{Q}$.

4.3. *Constructing the Semi-conjugacy.* We are now ready to construct the semi-conjugacy $f\colon \mathcal{A} \to \mathcal{M}(P, <)$. We will use the transit time functions τ_p and the Lyapunov function Λ as the coordinates of the function. We define f piecewise. If $x \in S_p$, define $f(x) = p$, the corresponding vertex in $\mathcal{M}(P, <)$. If $x \in C(S_p, S_q)$, there is some maximal totally ordered sequence $p_0 < p_1 < \cdots < p_n$ that contains P_x. The trajectory of x will be mapped into the simplex $[p_0 p_1 \cdots p_n]$ via the composition

$$C(S_p, S_q) \xrightarrow{F_{pq}} [0, n] \times I^{n-1} \xrightarrow{\eta} Q \xrightarrow{\lambda} \Delta^n,$$

where

$$F_{pq} = (\Lambda(x), 0, ..., 0, \tan^{-1}(\tau_p(x)), ..., \tan^{-1}(\tau_q(x)), 0, ..., 0).$$

This defines the semi-conjugacy of Theorem 1.1. We must show that f is well-defined, continuous, and a semi-conjugacy.

Proof of Theorem 1.1. The first observation is that $f(x)$ is independent of the sequence $p_0 < \cdots < p_n$ chosen to extend P_x. If $x \in C(S_p, S_q)$, then $\tau_p(x) = \tau_q(x) = \infty$, and $\tau_r(x)$ is finite for all $q < r < p$. Thus $l(F_{pq}(x)) = q$ and $r(F_{pq}(x)) = p$. Further, Λ decreases from $n(p)$ to $n(q)$ along the orbit, while the transit time functions are constant along the orbit. Thus f maps $x \cdot R$ into an orbit in the interior of the simplex spanned by $\{r \in P \mid q \leqslant r \leqslant p\}$. That is, $C(S_p, S_q)$ is mapped to $C(M_p, M_q)$.

To show that f is continuous, consider $x \in C(S_p, S_q)$. If $x_n \to x$, then $\Lambda(x_n) \to \Lambda(x)$, and for n sufficiently large, $P_x \subset P_{x_n}$, with $\tau_p(x_n) \to \tau_p(x)$ for all $p \in P_x$. Thus, if we consider the non-trivial coordinates of $F_{pq}(x)$, we see convergence. The coordinates $\tau_r(x_n)$ for $r < q$ or $r > p$ may not converge to 0, but these coordinates are collapsed out under the identification η, so the composition $\eta \circ F$ is continuous, and so f is continuous.

Since the transit time functions are constant on orbits in $\mathscr{A}$, the orbit $x \cdot R$ maps under F_{pq} to the orbit $(n(q), n(p)) \times \{t_r = \tan^{-1}(\tau_r(x))\}$. The reparameterization of the previous section insures that, if $F_{pq}(x) = F_{pq}(y)$, then $F_{pq}(\phi'(x, t)) = F_{pq}(\phi'(y, t))$. Another time reparameterization of the flow on $\mathscr{A}$ is required to make f a semi-conjugacy. Since f maps orbits to orbits, and is one-to-one on orbits, for every $(x, t) \in \mathscr{A} \times R$, there is an $s(t)$ such that $f \circ \phi'(x, s(t)) = \psi(f(x), t)$. The previous reparameterization of the flow on $\mathscr{A}$ insures that the value $s(t)$ depends only on $f(x)$ and not on x itself. That value is arbitrary if $f(x)$ lies in one of the Morse sets, and is unique otherwise. The continuity of f and the flows guarantees the continuity of s on $\mathscr{A} \setminus \bigcup_{p \in P} S_p$. Take any continuous extension of s to all of $\mathscr{A}$. We can thus reparameterize the flow on $\mathscr{A}$ by $\phi''(x, t) = \phi'(x, s(t))$, and with this reparameterization, f is a semi-conjugacy.

5. SURJECTIVITY

Having constructed f, it remains only to show that it is surjective. The construction of f makes it clear that $f^{-1}(M_p) = S_p$, and that $f^{-1}(C(M_p, M_q)) = C(S_p, S_q)$. Since each M_p is a point, it is trivial that f maps onto all M_p. Thus, to show f is surjective, we only need to show that it maps onto each $C(M_p, M_q)$. It is at this point that we require the hypothesis (**H4**). We require this because the "sphericity" provides an algebraic test for surjectivity: if $f\colon f^{-1}(M(A_p)) \to M(A_P)$ is not surjective, then $f_*\colon H_{n(p)-1}(f^{-1}(M(A_p))) \to H_{n(p)-1}(M(A_p))$ is trivial. A preliminary to establishing the surjectivity of f is then to determine the behavior of f_*.

We first prove that f_{p*} is an isomorphism for every $p \in P$. We proceed inductively on $n(p)$. For $n(p) = 0$, we have $L_p = \varnothing$, so f_{p*} is simply $H_0(f^{-1}(N_p)) \to H_0(N_p)$, which is clearly an isomorphism. For $n(p) > 0$, choose $q<p$ with $n(q)=n(p)-1$. There is then a commutative diagram [11]

$$\begin{array}{ccccccc} 0 & \longrightarrow & CH_{n(p)}(S_p) & \xrightarrow{\partial_{qp}} & CH_{n(p)-1}(S_q) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_{p*}} & & \downarrow{\scriptstyle f_{q*}} & & \\ 0 & \longrightarrow & CH_{n(p)}(M_p) & \xrightarrow{\partial_{Mqp}} & CH_{n(p)-1}(M_q) & \longrightarrow & 0 \end{array}$$

By hypothesis (**H3**), the induction hypothesis and Proposition 3.16, the homomorphisms ∂_{qp}, f_{q*} and ∂_{Mqp} are all isomorphisms. Clearly, f_{p*} is also an isomorphism. Since the entries ∂_{qp} and ∂_{Mqp} are the only non-zero entries of $\Delta(P)$ and $\Delta_M(P)$, it follows that $\Delta_M(P)\, F=F\Delta(P)$.

To prove that $f_*(I)$ is an isomorphism for all intervals I in P, we now proceed by induction on $|I|$. If $|I|>1$, choose a maximal element $p\in I$ and let $J=I\backslash\{p\}$. Then $(M(J), M_p)$ is an attractor-repeller pair for $M(I)$ and $(S(J), S_p)$ is an attractor-repeller pair for $S(I)$. These indices are related by the attractor-repeller sequence diagram

$$\begin{array}{ccccc} CH_{n(p)+1}(S_p) & \xrightarrow{[\Delta(J,p)]} & CH_{n(p)}(S(J)) & \longrightarrow & CH_{n(p)}(S(I)) \\ \downarrow{\scriptstyle f_{p*}} & & \downarrow{\scriptstyle f_*(J)} & & \downarrow{\scriptstyle f_*(I)} \\ CH_{n(p)+1}(M_p) & \xrightarrow{[\Delta_M(J,p)]} & CH_{n(p)}(M(J)) & \longrightarrow & CH_{n(p)}(M(I)) \end{array}$$

By hypothesis, f_{p*} and $f_*(J)$ are isomorphisms, so the Five Lemma implies that $f_*(I)$ is as well.

It suffices to show that if $n(p)>0$, then $W^u(S_p)$ maps onto $W^u(M_p)$. From Lemma 3.11, we can choose an index pair (N_p, L_p) for M_p such that N_p is contractible and L_p is a cross-section of the flow. In particular, L_p contains a cross-section C_p of $W^u(M_p)$, and there is a strong deformation retraction $h: L_p\times I\to L_p$ onto C_p. From the simplicial structure, h can be constructed so that, for every $x\in C_p$, proposition$_1(h^{-1}(x))$ is invariant under h (i.e., if $h(y,1)=x$, then $h(h(y,s),1)=x$ for all $0<s<1$.

Since $(f^{-1}(N_p), f^{-1}(L_p))$ is an index pair for S_p, the map f_{p*}: $CH_*(S_p)\to CH_*(M_p)$ is represented by

$$\begin{array}{ccc} H_{n(p)}(f^{-1}(N_p), f^{-1}(L_p)) & \longrightarrow & H_{n(p)-1}(f^{-1}(L_p)) \\ \cong\downarrow{\scriptstyle f_{p*}} & & \downarrow{\scriptstyle f_*} \\ H_{n(p)}(N_p, L_p) & \xrightarrow{\cong} & H_{n(p)-1}(L_p) \end{array}$$

Clearly, this implies that $f_*: H_{n(p)-1}(f^{-1}(L_p))\to H_{n(p)-1}(L_p)$ is surjective.

If f does not map onto C_p, then there is some point $x\in C_p$ and a neighborhood U of x such that the compact set $f(\mathscr{A})$ is disjoint from U.

We can choose L_p sufficiently close to C_p that $H^{-1}(x) \subset U$. Since L_p\proposition$_1(h^{-1}(x))$ deforms onto $C_p \backslash \{x\}$, it is contractible. But this implies that $f_*: H_{n(p)-1}(f^{-1}(L_p)) \to H_{n(p)-1}(L_p)$ factors through a trivial homology group, and so cannot be surjective.

6. CONCLUSION

As discussed in the introduction, one of the goals of this work was to gain some insight into the Conley index information required to construct a model flow and semi-conjugacy. Now that we have established that (**H0**)–(**H4**) imply such a construction, it is natural to turn our attention to those hypotheses. In particular, we would like to understand how verifiable they are in practice; and if they are necessary conditions for the construction. We will also consider possible generalizations of the construction.

6.1. *Verifying the Hypotheses.* To apply this theorem to an attractor $\mathscr{A}$, we must be able to carry out the following computations:

(1) Isolate the attractor in X.

(2) Determine that $\mathscr{A}$ admits a Morse decomposition with index set P.

(3) Compute the homology Conley index of each of the Morse sets.

(4) Compute a connection matrix for the Morse decomposition.

We must further show that the objects identified satisfy the following conditions:

(5) The Morse sets must all have the homology Conley index of hyperbolic fixed points.

(6) All non-zero entries in the connection matrix must be isomorphisms.

(7) If $\Delta_{qp} \neq 0$, then p and q are not adjacent in the flow-defined order.

At this point, the model can be constructed, and the last required condition can be tested:

(8) Each $M(A_p)$ must be homeomorphic to $S^{n(p)-1}$.

These eight steps have varying degrees of difficulty associated with them. Assuming the first seven steps have been carried out, the last step is straightforward (with Theorem 3.15 available to assist). Similarly, if the first three steps have been carried out, verifying (5) and (6) is trivial. Thus, the only steps of any substance are the first four (computing the Conley index information) and the seventh (verifying that the Conley index information has detected all connecting orbits). It is important to note that (7) is fundamentally different than the first four. The first four are purely computational issues, while (7) concerns the ability of those computations to detect the essential dynamical behavior.

The computational issues are considerably easier to deal with, as one of the strengths of the Conley index is its computability. Detecting an attractor and a Morse decomposition, computing the indices of the Morse sets and computing a connection matrix are all well-understood processes. Typically, an attractor is detected by finding a positively invariant neighborhood; a Morse decomposition is detected by a Lyapunov function; homology indices are computed by continuation; and connection matrices are computed by the algebraic relations of the attractor-repeller exact sequences. Moreover, the ongoing development of computer-aided Conley index computations [8, 17, 19, 20, 24] promises to make all of these calculations even more tractable, even in cases when the system is only known from experimental data [18].

The real issue, then, is the verification that S_p and S_q are not adjacent if $\Delta_{qp} = 0$. This is emblematic of a much deeper question: does the algebraic information of the Conley index faithfully reflect the dynamical structure of the original system. Clearly, the index information itself cannot answer such a question. Some other form of analysis is required. For these results to be of any practical value, we must be able to carry out that analysis with only partial knowledge of the system. Fortunately, the condition we seek to verify is a negative one: showing that, if $\Delta_{qp} = 0$, then S_p and S_q are *not* adjacent in the flow-defined order. That is, either there is some r with $q < r < p$, or $p < q$, or p and q are unrelated in the partial order. There are a variety of ways this can be done.

- If $\Delta_{pq} \neq 0$, then $p < q$, so $q \nless p$.
- If there is an explicitly given Lyapunov function $L \colon \mathscr{A} \to R$ and $L(S_p) < L(S_q)$, then there can be no connection from S_p to S_q.
- If $n(p) - n(q) > 2$ and there are $p_1, ..., p_k$ with $\Delta_{qp_1} \Delta_{p_1 p_2} \cdots \Delta_{p_k p} \neq 0$, then $q < p_1 < \cdots < p_k < p$, so p and q are not adjacent.
- If all else fails, we must estimate $W^u(S_p)$ and $W^s(S_q)$, and show that $W^u(S_p) \cap W^s(S_q) = \varnothing$.

In principle, this is the type of calculation that can be performed numerically, and made rigorous by error estimates. While not an easy matter, such calculations are feasible, particularly if an explicit Lyapunov function is given. The multi-valued map techniques now being developed to carry out the index computations [8, 17, 19, 20, 24] may also be used in these calculations.

Once the partial order $(P, <)$ has been identified, the construction of $\mathscr{M}(P, <)$ proceeds in a purely routine fashion. While Theorem 3.15 does not give a purely graph-theoretic condition for (**H4**), it does provide tests for (**H4**) to hold, or to fail. Alternatively, once $\mathscr{M}(P, <)$ is constructed, the verification of **H4** from $\mathscr{M}(P, <)$ is straightforward.

6.2. *Necessity of the Hypotheses.* The conditions are not strictly necessary, in the sense that there are examples in which some or all of hypotheses (**H0**)–(**H4**) are not satisfied, but the conclusions of theorems 1.1 and 1.2 hold. However, there are also examples that make it clear that some hypotheses of this type are required. In this section, we examine some of these examples and counter-examples. Of course, without hypothesis (**H1**), the construction is not even defined, so we limit our concern to the other four hypotheses.

First, the invariant set need not be an attractor. Take any compact manifold N with a Morse function. The critical points form a Morse decomposition which satisfies (**H0**)–(**H2**). If we limit our attention to a manifold and Morse function that satisfy (**H3**) and (**H4**), then there is a semi-conjugacy from N to a model system $\mathscr{M}(P, <)$. Now, embed N as $N \times \{0\}$ in $N \times \mathbb{R}^k$, and take a product flow such that $\{0\}$ is repelling in $\mathbb{R}^k$. Clearly, N is no longer an attractor in $N \times \mathbb{R}^k$, yet the semi-conjugacy still exists. Of course, it no longer produces an isomorphism on the Conley indices. If we retain the requirement that the Conley indices are isomorphic, then $\mathscr{A}$ must be an attractor in the ambient space X, since $\mathscr{M}(P, <)$ is certainly an attractor in itself.

The hypothesis (**H2**) is very strong, and there is certainly no reason to expect it to be a necessary condition for the construction of a model and a semi-conjugacy. Indeed, the original paper [12] constructed a model for a system with Morse sets that have the Conley index of a hyperbolic periodic orbit. While that example shows that it is not necessary to assume that Morse sets have the homology Conley index of hyperbolic fixed points, it also suggests why it is natural to make such an assumption.

If S_p has a more complicated homology index, we must decide between (at least) two alternatives. On the one hand, we can employ the construction of $\mathscr{M}(P, <)$ used here, which collapses each S_p to a point. On the other hand, we may seek to use the homology index to "guess" the appropriate model M_p for S_p, then build the total model $\mathscr{M}$ by collating these model Morse sets. That was the strategy employed in [12]. There we hypothesized that whenever S_p had the homology Conley index of a hyperbolic periodic orbit, it had a return map defined on a neighborhood. With this information, it was natural to take a single periodic orbit as the model M_p.

Obstructions to generalizing this approach are:

- How do we recognize from the homology index what the underlying space should be?
- How do we know what flow to put on that space?
- Δ_{qp} may now be unreliable as a guide to whether or not S_p and S_q are adjacent.
- Δ_{qp} is now a matrix, so there are many different ways that it can be non-zero. How do we interpret these dynamically?
- How do we assemble the model Morse sets to form $\mathscr{M}$?
- How do we put co-ordinates on $\mathscr{M}$ so that we can construct the semi-conjugacy?

These obstructions are substantial, and it is not clear that there is any general construction that will successfully deal with all of them. Certainly, [12] suggests that there will be at least some cases that are tractable. Assumptions such as *-hyperbolicity [3] may help to expand that collection. However, if the Morse sets are assumed to have the homology Conley index of a hyperbolic periodic orbit, these obstructions (for the most part) vanish. Obviously, (**H2**) is not enough to eliminate all difficulties, hence the need for (**H3**) and (**H4**). We now turn to a consideration of those hypotheses.

As discussed above (**H3**) contains the crucial assumption that the algebra of the Conley index detects all connections. To see that this assumption need not always be satisfied, consider the attractor-repeller decomposition of the circle shown in Fig.8(a). The Morse sets are hyperbolic fixed points with $n(i)=i$. Since the index of the total invariant set S is the direct sum of the indices of the Morse sets, the connection matrix must be trivial. That is, the two branches separately have connection homomorphisms that are isomorphisms, but they have opposite orientations and so cancel one another. The algebra provides no evidence of any connections between S_1 and S_0. Similarly, in a situation in which $W^u(R)$

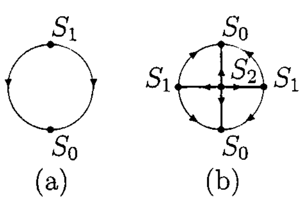

FIG. 8. Flows on the circle and projective plane.

and $W^s(A)$ intersect non-transversely, the connection homomorphism can be trivial.

Assumption (**H3**) assumes more than just that Δ_{qp} is non-zero when S_p and S_q are adjacent. It assumes that Δ_{qp} is either an isomorphism or is trivial. This need not always occur. Consider the flow on $\mathbb{R}P^2$ generated from the flow in Fig. 8(b) formed by identifying antipodal points on the boundary. Each of the Morse sets S_0, S_1, S_2 is hyperbolic with $n(i)=i$. The connection matrix must compute the homology of $\mathbb{R}P^2$ from the chain complex

$$C_i \Delta(P) = \begin{cases} \mathbb{Z}, & i=0,1,2 \\ 0, & \text{otherwise} \end{cases}.$$

Clearly, the unique matrix that does this is

$$\Delta(P) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

The first example (i.e., adjacent entries with trivial algebra) appears at this point to be an essential obstruction. If the algebra carrying the dynamical information, there is no reason to expect a model based on the algebra to be meaningful. The second example (i.e., non-trivial entries in $\Delta(P)$ that are not isomorphisms) suggests that a more general construction of the model space may be needed. Suppose we retain (**H2**), and weaken (**H3**) to

(**H3′**) There is a unique connection matrix $\Delta(P)$. This matrix has the property that Morse sets S_p and S_q are adjacent in the flow-defined ordering if and only if the connection matrix entry Δ_{qp} is non-zero.

If we define $C_n = \bigoplus_{p \in P_n} CH_n(S_p)$ and $\partial_n = \Delta(P_{n-1}, P_n)\colon C_n \to C_{n-1}$, then it is natural to interpret the chain complex $\{C_n, \partial_n\}$ as the cellular chains of a CW-complex. That is, we might try to construct a CW model instead of a simplicial model for the flow. This is hardly a new idea. After all, Morse theory describes a CW decomposition of a manifold. But, in the Morse theory setting, we start with the assumption of a flow on a manifold. Here, we are starting with an unknown attractor, that *looks like* a Morse flow on the homology level. Can we, from homological data that emulates that of a Morse flow, construct an actual Morse flow and a semi-conjugacy onto it? This is an open question at present, and will be the subject of future investigations.

Finally, we turn to (**H4**). Example 3.14 shows that $M(A_p)$ need not have the homology of $S^{n(p)-1}$, and so the homology Conley indices of M_p and

S_p need not be isomorphic. Some hypothesis of this type is needed. But, could it suffice to assume that $M(A_p)$ is a homology sphere, or a homotopy sphere, to prove that f is surjective? Is the isomorphism of indices required at all for f to be surjective? These are open questions at this point.

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